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HARMONIC ANALYSIS OF THE TWO-DIMENSIONAL FLOW OF AN INCOMPRESSIBLE VISCOUS FLUID*

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Introduction. In this paper, we consider the two dimensional flow of an incompressible viscous fluid

a) in a *bounded domain* D, the velocity vanishing at the boundary B of the domain; B is assumed to be a regular curve;[1]

b) in the entire plane, the velocity vanishing at infinity, this case being regarded as a limiting case of a).

Since the fluid is assumed to be *incompressible*, there exists a *stream function* $\psi(x, y, t)$ such that the *two components* u and v of the velocity are given by

$$u(x, y, t) = \frac{\partial \psi}{\partial y}, \tag{1}$$

$$v(x, y, t) = -\frac{\partial \psi}{\partial x}. \tag{2}$$

According to the *boundary conditions*, we have

$$u(x, y, t) = 0, \qquad v(x, y, t) = 0 \qquad \text{on } B \tag{3}$$

or

$$\psi(x, y, t) = 0, \qquad \frac{\partial \psi}{\partial n} = 0 \qquad \text{on } B. \tag{4}$$

The *vorticity* is given by[2]

$$\zeta(x, y, t) = -\tfrac{1}{2}\Delta\psi. \tag{5}$$

According to the Navier-Stokes equations the vorticity must satisfy the equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu\Delta\zeta \tag{6}$$

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[1] What is meant here by the term *regular curve* is only that B fulfills the conditions required for the use of the Green formula.

[2] $\Delta\psi$ indicating the Laplacian: $\Delta\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2$.

in which ν is a constant, the *kinematic viscosity* of the fluid. Substituting the values (1), (2), and (5) into (6), we obtain a partial (non-linear) differential equation for $\psi(x, y, t)$, which is characteristic for the two dimensional flow of an incompressible

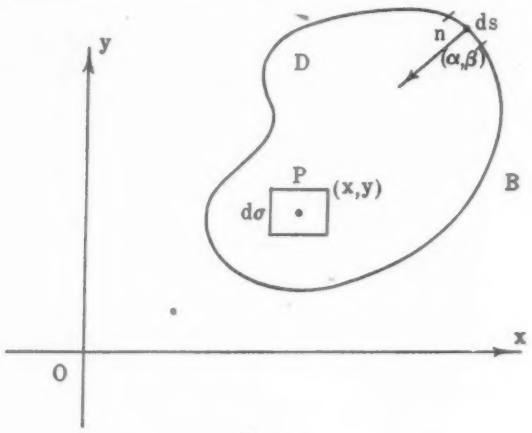

FIG. 1

viscous fluid. We shall only deal with *regular flows*, that is with stream functions such that *the functions*

$$\psi,\ u,\ v,\ \zeta,\ \frac{\partial\zeta}{\partial x},\ \frac{\partial\zeta}{\partial y},\ \Delta\zeta,\ \frac{\partial\psi}{\partial t},\ \frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial\zeta}{\partial t}$$

are continuous in x, y, t for (x, y) in $D+B$ and $t_1 \leqq t \leqq t_2$.

This paper is divided in three parts. In the first part, the Fourier transforms $\Psi(\omega_1, \omega_2, t)$, $U(\omega_1, \omega_2, t)$, $V(\omega_1, \omega_2, t)$, $Z(\omega_1, \omega_2, t)$ of the stream function ψ, the velocity components u, v, and the vorticity ζ are introduced and it is proved that Ψ, U and V are obtained from Z by multiplication by simple rational functions of ω_1 and ω_2. Next the kinetic energy of the flow

$$E = \frac{1}{2}\int_D (u^2 + v^2)d\sigma, \qquad (d\sigma = dxdy) \tag{7}$$

and its *spectral decomposition*

$$E = \int_\Omega \gamma(\omega_1, \omega_2, t)d\omega, \qquad (d\omega = d\omega_1 d\omega_2) \tag{8}$$

are considered and it is shown that the *spectral function* $\gamma(\omega_1, \omega_2, t)$ has also a very simple expression in Z. In other words, the entire harmonic analysis of the flow can be based on the Fourier transform of the vorticity.

This part of the paper uses only the definitions of ψ, u, v, ζ and the boundary condition; the results are valid whether the fluid is viscous or not.

In the second part Eq. (6) is used and some inequalities are given which must be satisfied by the Fourier transforms when the variation of the time t is considered. Some of these inequalities concern the variation of the spectral function with time.

All inequalities are based on the fact that *the kinetic energy of the flow is decreasing more rapidly than an exponential function of t.*

Next, an integrodifferential equation for the Fourier transform Z of the vorticity is derived from (6). This equation might be the starting point for the study of the variation of the spectral function with the time t.

Finally, in the third part, the case that the flow fills the entire plane

$$X \qquad -\infty < x < +\infty, \qquad -\infty < y < +\infty$$

is considered as the limit of the flow in a bounded domain, the fluid being at rest at infinity. The integrodifferential equation simplifies considerably in this case.

If $f(x, y)$ is a real function of the real variables x and y defined[3] and continuous in $D+B$, its *Fourier transform* is given by

$$F(\omega_1, \omega_2) = \frac{1}{4\pi^2}\int_D f(x, y)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{9}$$

where the *frequencies* ω_1, ω_2 are real. We shall use the following well-known properties of the Fourier transforms.[4]

a) $F(\omega_1, \omega_2)$ is a complex function of the two real variables ω_1, ω_2 defined in the entire plane

$$\Omega \qquad -\infty < \omega_1 < +\infty, \qquad -\infty < \omega_2 < +\infty.$$

b) $$F(-\omega_1, -\omega_2) = \overline{F}(\omega_1, \omega_2),$$

where $\overline{F}$ denotes the conjugate of F.

c) $F(\omega_1, \omega_2)$ is a continuous function of the two variables ω_1, ω_2 in every point of Ω.

d) $F(\omega_1, \omega_2)$ is bounded in Ω:

$$|F(\omega_1, \omega_2)| \leqq \frac{1}{4\pi^2}\int_D |f(x, y)|\, d\sigma. \tag{10}$$

We shall also use the bound

$$|F(\omega_1, \omega_2)| \leqq \frac{\sqrt{S}}{4\pi^2}\left[\int_D f(x, y)^2 d\sigma\right]^{1/2}, \tag{11}$$

where $S=\int_D d\sigma$. Equation (11) is deduced from (10) by means applying Schwarz' inequality.

e) If $|\omega_1|+|\omega_2|\to+\infty$, then $F(\omega_1, \omega_2)\to 0$ (Riemann's theorem).

f) If $\phi(x, y)=af(x, y)+bg(x, y)$, where a, b are constants, and $f(x, y)$, $g(x, y)$ continuous functions in $D+B$ having the Fourier transforms $F(\omega_1, \omega_2)$, $G(\omega_1, \omega_2)$, then $\phi(x, y)$ has the Fourier transform $\Phi(\omega_1, \omega_2)=aF(\omega_1, \omega_2)+bG(\omega_1, \omega_2)$.

g) If $f(x, y)$ and $g(x, y)$ are continuous functions in $D+B$ having the Fourier transforms $F(\omega_1, \omega_2)$ and $G(\omega_1, \omega_2)$, respectively, and if

$$F(\omega_1, \omega_2) = G(\omega_1, \omega_2),$$

[3] In order to apply directly most of the known results of the theory of the Fourier transforms, it is often useful to consider $f(x, y)$ as defined in the entire plane X with $f(x, y)=0$ at every point outside $D+B$.

[4] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, pp. 183–197.

then

$$f(x, y) = g(x, y)$$

in every point of $D+B$.

h) If $F(\omega_1, \omega_2)$ is *absolutely integrable*,

$$\int_\Omega |F(\omega_1, \omega_2)| \, d\omega < +\infty,$$

the integral

$$\int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega$$

defines a continuous function of x, y for all values of x, y; in $D+B$:

$$f(x, y) = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega,$$

while

$$0 = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega$$

outside $D+B$. Since the integral is continuous for all values of x, y, the function $f(x, y)$ must vanish on B. Thus, if the given function $f(x, y)$ does not vanish on B, its Fourier transform cannot be absolutely integrable.[5]

i) If $f(x, y)$ has continuous derivatives $\partial f/\partial x$, $\partial f/\partial y$ in $D+B$ (which is always the case for the functions considered here), one has

$$f(x, y) = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega \tag{12}$$

in every point of D, the integral being now an *improper integral* defined as the limit

$$\lim_{\lambda \to +\infty} \int_{C_\lambda} F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega, \tag{13}$$

C_λ being the circle $\omega_1^2 + \omega_2^2 \leqq \lambda^2$ (Cauchy's principal value). As a rule, (12) does not hold on the boundary B.

j) $f(x, y)$ and $g(x, y)$ being continuous functions with continuous derivatives in $D+B$ and $F(\omega_1, \omega_2)$ and $G(\omega_1, \omega_2)$ their Fourier transforms, we have

$$\frac{1}{4\pi^2} \int_D f(x, y) g(x, y) e^{-i(\theta_1 x + \theta_2 y)} d\sigma = \int_\Omega F(\omega_1, \omega_2) \overline{G}(\omega_1 + \theta_1, \omega_2 + \theta_2) d\omega, \tag{14}$$

the meaning of the integral being the same as in (12). Here, θ_1 and θ_2 are arbitrary real variables; in particular for $\theta_1 = \theta_2 = 0$ one has Parseval's formula:

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[5] For instance, taking for $D+B$ the square $-1 \leqq x \leqq +1$, $-1 \leqq y \leqq +1$ and assuming that $f(x, y) = 1$ in $D+B$ we obtain the Fourier transform $F(\omega_1, \omega_2) = (1/\pi^2 \omega_1 \omega_2) \sin \omega_1 \sin \omega_2$ which is not absolutely integrable. Eq. (12) holds, however, with the definition (13).

$$\frac{1}{4\pi^2}\int_D f(x, y)g(x, y)d\sigma = \int_\Omega F(\omega_1, \omega_2)\overline{G}(\omega_1, \omega_2)d\omega. \tag{15}$$

1. Let us introduce the Fourier transforms of the stream function, the velocity components and the vorticity:

$$\Psi(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \psi(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{16}$$

$$U(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D u(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{17}$$

$$V(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D v(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{18}$$

$$Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \zeta(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma. \tag{19}$$

The first part of the present paper deals with the purely kinematical significance of ψ, u, v, ζ, all computations being supposed to be made at a given time t. Thus, there is no particular need for stressing the particular value of t. For the sake of brevity we shall therefore write $\psi(x, y)$, $\Psi(\omega_1, \omega_2)$, etc. for $\psi(x, y, t)$, $\Psi(\omega_1, \omega_2, t)$, etc.

THEOREM I—The Fourier transforms of the stream function and of the velocity components are expressed in terms of the Fourier transform of the vorticity by means of

$$\Psi(\omega_1, \omega_2) = \frac{2}{\omega_1^2 + \omega_2^2}Z(\omega_1, \omega_2), \tag{20}$$

$$U(\omega_1, \omega_2) = \frac{2i\omega_2}{\omega_1^2 + \omega_2^2}Z(\omega_1, \omega_2), \tag{21}$$

$$V(\omega_1, \omega_2) = \frac{-2i\omega_1}{\omega_1^2 + \omega_2^2}Z(\omega_1, \omega_2). \tag{22}$$

Substituting (1) into (17) we obtain

$$U = \frac{1}{4\pi^2}\int_D \frac{\partial\psi}{\partial y}e^{-i(\omega_1 x+\omega_2 y)}d\sigma$$

$$= \frac{1}{4\pi^2}\int_D \frac{\partial}{\partial y}[\psi e^{-i(\omega_1 x+\omega_2 y)}]d\sigma + \frac{i\omega_2}{4\pi^2}\int_D \psi e^{-i(\omega_1 x+\omega_2 y)}d\sigma.$$

Application of Green's formula to the first integral yields

$$U = -\frac{1}{4\pi^2}\int_B \beta\psi e^{-i(\omega_1 x+\omega_2 y)}ds + i\omega_2\Psi.$$

On account of the boundary condition (4) the first integral equal zero. Thus

$$U = i\omega_2\Psi. \tag{23}$$

Similarly

$$V = -i\omega_1\Psi. \tag{24}$$

Next, let us apply Green's formula

$$\int_D (f\Delta g - g\Delta f)d\sigma = -\int_B \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)ds$$

setting $f=\psi$, $g=e^{-i(\omega_1 x+\omega_2 y)}$. On account of the boundary condition (4) we have:

$$\int_D [\psi\Delta e^{-i(\omega_1 x+\omega_2 y)} - \Delta\psi e^{-i(\omega_1 x+\omega_2 y)}]d\sigma = 0.$$

If we replace $\Delta\psi$ by -2ζ in accordance with (5), and remark that

$$\Delta e^{-i(\omega_1 x+\omega_2 y)} = -(\omega_1^2+\omega_2^2)e^{-i(\omega_1 x+\omega_2 y)}, \tag{25}$$

we obtain

$$-(\omega_1^2+\omega_2^2)\int_D \psi e^{-i(\omega_1 x+\omega_2 y)}d\sigma + 2\int_D \zeta e^{-i(\omega_1 x+\omega_2 y)}d\sigma = 0$$

which is Eq. (20). Inserting the value (20) of Ψ into (23) and (24) we immediately obtain the expression (21) and (22) and our theorem is proved.[6]

Combining (21) and (22) we obtain the following relation

$$2iZ = \omega_2 U - \omega_1 V. \tag{26}$$

THEOREM II—The spectral function $\gamma(\omega_1, \omega_2)$ of the kinetic energy E which is defined by (8), *is expressed in terms of the Fourier transform of the vorticity by means of*

$$\gamma(\omega_1, \omega_2) = 8\pi^2 \frac{|Z(\omega_1, \omega_2)|^2}{\omega_1^2+\omega_2^2}. \tag{27}$$

To prove this, we start from Green's formula

$$\int_D \psi\Delta\psi d\sigma + \int_D \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right]d\sigma = -\int_B \psi\frac{\partial\psi}{\partial n}ds.$$

The integral over B equals zero on account of the boundary condition (4). If we replace $\Delta\psi$ by its value (5) and $\partial\psi/\partial x$, $\partial\psi/\partial y$ by (1) and (2), we obtain

$$E = \int_D \psi\zeta d\sigma. \tag{28}$$

Applying Parseval's formula (15), we find

$$E = 4\pi^2 \int_\Omega \Psi\overline{Z}d\omega, \tag{29}$$

and hence

$$\gamma(\omega_1, \omega_2) = 4\pi^2\Psi(\omega_1, \omega_2)\overline{Z}(\omega_1, \omega_2). \tag{30}$$

[6] It is perhaps worthwhile to stress that the boundary condition (4) plays an essential part in this proof. As a rule, the relation $F_1=i\omega_1 F$ does not hold between the Fourier transforms $F(\omega_1, \omega_2)$ and $F_1(\omega_1, \omega_2)$ of $f(x, y)$ and $\partial f/\partial x$. For instance for the function $f=1$ considered in remark 5, $F_1=0$, while $i\omega_1 F=(i\pi^2/\omega_2)\sin\omega_1\sin\omega_2$.

From this we deduce (27) by inserting the value (20) for Ψ.

The theorems I and II reduce the harmonic analysis of the flow to the study of the Fourier transform Z of the vorticity. We shall now establish some properties of this function. According to (12).

$$\zeta(x, y) = \int_{\Omega} Z(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega$$

$$= \lim_{\lambda \to +\infty} \int_{C_\lambda} Z(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 y)} d\omega. \tag{31}$$

We can then interpret $Z(\omega_1, \omega_2) d\omega_1 d\omega_2$ as the contribution of vortices, the frequencies of which are between ω_1 and $\omega_1 + d\omega_1$, and ω_2 and $\omega_2 + d\omega_2$.

$Z(\omega_1, \omega_2)$ is continuous and bounded in the whole plane Ω. Noting that

$$\Psi(0, 0) = \frac{1}{4\pi^2} \int_D \psi d\sigma$$

is finite, we obtain from (20),

$$Z(0, 0) = 0. \tag{32}$$

Equation (26) then yields

$$2|Z| \leqq |\omega_2||U| + |\omega_1||V|. \tag{33}$$

On account of (11) we have

$$|U| \leqq \frac{\sqrt{S}}{4\pi^2} \left[\int_D u^2 d\sigma \right]^{1/2} \quad |V| \leqq \frac{\sqrt{S}}{4\pi^2} \left[\int_D v^2 d\sigma \right]^{1/2},$$

and consequently

$$|U|^2 + |V|^2 \leqq \frac{S}{8\pi^2} E. \tag{34}$$

Equation (33) gives then the following upper bound for $|Z|$:

$$|Z| \leqq \frac{\sqrt{2S}}{8\pi^2} (|\omega_1| + |\omega_2|) \sqrt{E}. \tag{35}$$

At this point it seems interesting to note a property of the Fourier transform Z. *If $\phi(x, y)$ is an arbitrary harmonic function in D, and $\Phi(\omega_1, \omega_2)$ is its Fourier transform, Z is orthogonal to Φ*, i.e.,

$$\int_{\Omega} Z(\omega_1, \omega_2) \overline{\Phi}(\omega_1, \omega_2) d\omega = 0.$$

In fact, the vorticity ζ is orthogonal to any harmonic function ϕ in D.[7] To prove this we need only use Green's formula

[7] J. Kampé de Fériet, *On a property of the Laplacian of a function in a two dimensional bounded domain, when the first derivatives of the function vanish at the boundary*, Mathematics Magazine No. 2, 1947.

$$\int_D (\phi\Delta\psi - \psi\Delta\phi)d\sigma = -\int_B \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds.$$

On account of the boundary condition (4) the second integral vanishes. Setting

$$\Delta\psi = -2\zeta, \qquad \Delta\phi = 0$$

in the first integral we find

$$\int_D \phi\zeta d\sigma = 0$$

which, on account of (15), is fully equivalent to our statement.

Let us now consider the spectral function $\gamma(\omega_1, \omega_2)$. We can interpret $\gamma(\omega_1,\omega_2)d\omega_1 d\omega_2$ as the amount of kinetic energy coming from the vortices with frequencies between ω_1 and $\omega_1+d\omega_1$, and ω_2 and $\omega_2+d\omega_2$.

The Fourier transforms Ψ and Z being continuous and bounded in the entire plane Ω, we see from (30) that $\gamma(\omega_1, \omega_2)$ is also continuous and bounded in the entire plane. Moreover, $\gamma(\omega_1, \omega_2) \geqq 0$ for all values of ω_1 and ω_2; on account of the same formula

$$\gamma(0, 0) = 4\pi^2\Psi(0, 0)\overline{Z}(0, 0),$$

and hence

$$\gamma(0, 0) = 0. \tag{36}$$

More precisely, if we write Eq. (30) in the following manner

$$\gamma(\omega_1, \omega_2) = 2\pi^2(\omega_1^2 + \omega_2^2)\,|\,\Psi(\omega_1, \omega_2)\,|^2,$$

we see that

$$\lim_{\omega_1\to 0,\omega_2\to 0} \frac{\gamma(\omega_1, \omega_2)}{\omega_1^2 + \omega_2^2} = 2\pi^2\,|\,\Psi(0, 0)\,|^2. \tag{37}$$

Since Ψ and Z tend towards zero when $|\omega_1| + |\omega_2| \to +\infty$, we have

$$\lim_{|\omega_1|+|\omega_2|\to+\infty} \gamma(\omega_1, \omega_2) = 0. \tag{38}$$

From (27) it follows then that $\gamma(\omega_1, \omega_2)$ is decreasing more rapidly than $1/(\omega_1^2+\omega_2^2)$. Combining (27) and the inequality (35) we obtain an upper bound for $\gamma(\omega_1, \omega_2)$:

$$\gamma(\omega_1, \omega_2) \leqq \frac{S}{4\pi^2}\,\frac{(|\,\omega_1\,| + |\,\omega_2\,|)^2}{\omega_1^2 + \omega_2^2}\,E,$$

and hence

$$\gamma(\omega_1, \omega_2) \leqq \frac{S}{2\pi^2}\,E. \tag{39}$$

2. Let us now consider the variation of the flow with the time t, in accordance with Eq. (6) for a viscous incompressible fluid.

We shall use the following proposition:[8] *The kinetic energy is always decreasing*

[8] J. Kampé de Fériet, *Sur la décroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine plan borné*, C. R. Acad. Sci. Paris **223**, 1096–1098 (1946).

$(dE/dt<0)$; *more precisely, it is decreasing more rapidly than an exponential function of the time*:

$$E \leqq E_0 \exp\left(-\frac{8\nu}{\sqrt{K_D}}t\right). \tag{40}$$

Here, E_0 is the value of E at the time $t=0$, and K_D is a positive constant depending only on the domain D.

Applying this result to (35) and (39), we have the following

THEOREM III—The upper bounds of $|Z|$ *and* γ *given by the inequalities* (35) *and* (39) *are always decreasing, more precisely,*

$$|Z(\omega_1, \omega_2, t)| \leqq \frac{\sqrt{2SE_0}}{8\pi^2}(|\omega_1| + |\omega_2|)\exp\left(-\frac{4\nu}{\sqrt{K_D}}t\right) \tag{41}$$

$$|\gamma(\omega_1, \omega_2, t)| \leqq \frac{SE_0}{2\pi^2}\exp\left(-\frac{8\nu}{\sqrt{K_D}}t\right). \tag{42}$$

A more concrete picture is furnished by the surface $\gamma(\omega_1, \omega_2, t)$. This surface passes through the origin O and is asymptotic to the plane $\gamma=0$ for large values of ω_1, ω_2; its possible maxima are all below a given plane $\gamma = M$; with increasing time this plane approaches the plane $\gamma=0$ in an exponential manner.

In order to study the variations of the spectral function $\gamma(\omega_1, \omega_2, +)$ with the time t, we shall establish the equivalent of Eq. (6) in terms of Fourier transforms. To this end, we must compute the Fourier transforms of $\partial\zeta/\partial t$, of $u\partial\zeta/\partial x+v\partial\zeta/\partial y$ and of $\Delta\zeta$.

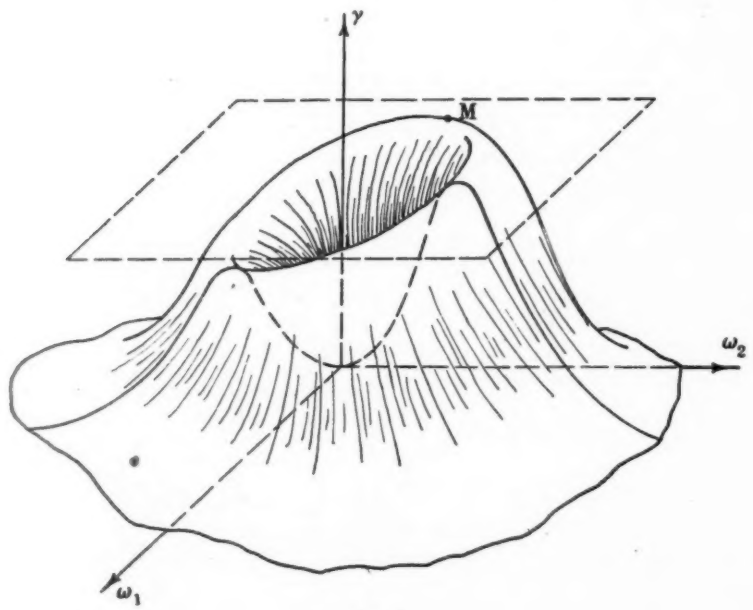

FIG. 2

For the first function the result is obvious. Since ζ and $\partial\zeta/\partial t$ are continuous in t, the functions Z and $\partial Z/\partial t$ are also continuous in t. Thus,

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \frac{\partial \zeta}{\partial t} e^{-i(\omega_1 x+\omega_2 y)} d\sigma. \tag{43}$$

The Fourier transform of $\partial\zeta/\partial t$ is therefore given by $\partial Z/\partial t$.

To compute the second Fourier transform, we start from the equation

$$\left(u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right) e^{-i(\omega_1 x+\omega_2 y)} = \frac{\partial}{\partial x}\left[u\zeta e^{-i(\omega_1 x+\omega_2 y)}\right] + \frac{\partial}{\partial y}\left[v\zeta e^{-i(\omega_1 x+\omega_2 y)}\right] + i(\omega_1 \zeta u + \omega_2 \zeta v) e^{-i(\omega_1 x+\omega_2 y)}.$$

We integrate this equation over D and note that by Green's formula,

$$\int_D \frac{\partial}{\partial x}\left[u\zeta e^{-i(\omega_1 x+\omega_2 y)}\right] d\sigma = -\int_B \alpha u \zeta e^{-i(\omega_1 x+\omega_2 y)} ds = 0, \tag{44}$$

$$\int_D \frac{\partial}{\partial y}\left[v\zeta e^{-i(\omega_1 x+\omega_2 y)}\right] d\sigma = -\int_B \beta v \zeta e^{-i(\omega_1 x+\omega_2 y)} ds = 0, \tag{45}$$

on account of the boundary condition (3). Accordingly,

$$\int_D \left(u\frac{\partial \zeta}{\partial y} + v\frac{\partial \zeta}{\partial y}\right) e^{-i(\omega_1 x+\omega_2 y)} d\sigma = i\omega_1 \int_D u\zeta e^{-i(\omega_1 x+\omega_2 y)} d\sigma + i\omega_2 \int_D v\zeta e^{-i(\omega_1 x+\omega_2 y)} d\sigma.$$

Using formula (14), we obtain

$$\frac{1}{4\pi^2}\int_D u\zeta e^{-i(\omega_1 x+\omega_2 y)} d\sigma = \int_\Theta U(\theta_1, \theta_2)\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2) d\theta,$$

$$\frac{1}{4\pi^2}\int_D v\zeta e^{-i(\omega_1 x+\omega_2 y)} d\sigma = \int_\Theta V(\theta_2, \theta_2)\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2) d\theta,$$

where the variables of integration are now called θ_1, θ_2, and $d\theta$ denotes $d\theta_1 d\theta_2$, the integral being extended over the entire plane Θ. We thus have proved that the Fourier transform of $u\partial\zeta/\partial x + v\partial\zeta/\partial y$ equals

$$i\int_\Theta \left[\omega_1 U(\theta_1, \theta_2) + \omega_2 V(\theta_1, \theta_2)\right]\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2) d\theta$$

or, what is the same by (21) and (22),

$$2\int_\Theta \frac{\theta_1\omega_2 - \theta_2\omega_1}{\theta_1^2+\theta_2^2} Z(\theta_1, \theta_2)\overline{Z}(\theta_1+\omega_1, \theta_2+\omega_2) d\theta. \tag{46}$$

Let us now consider the Fourier transform of the Laplacian of the vorticity:

$$C(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \Delta\zeta e^{-i(\omega_1 x+\omega_2 y)} d\sigma. \tag{47}$$

Applying Green's formula

$$\int_D [\Delta\zeta e^{-i(\omega_1 x+\omega_2 y)} - \zeta\Delta(e^{-i(\omega_1 x+\omega_2 y)})]d\sigma$$

$$= -\int_B \left[\frac{\partial\zeta}{\partial n} e^{-i(\omega_1 x+\omega_2 y)} - \zeta\frac{\partial}{\partial n}(e^{-i(\omega_1 x+\omega_2 y)})\right] ds$$

and setting

$$\Phi(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_B \left[\frac{\partial\zeta}{\partial n} + i(\omega_1\alpha + \omega_2\beta)\zeta\right] e^{-i(\omega_1 x+\omega_2 y)} ds, \tag{48}$$

we obtain

$$C(\omega_1, \omega_2, t) = -(\omega_1^2 + \omega_2^2)Z(\omega_1, \omega_2, t) - \Phi(\omega_1, \omega_2, t). \tag{49}$$

Reviewing the expressions of the Fourier transforms of $\partial\zeta/\partial t$, $u\partial\zeta/\partial x + v\partial\zeta/\partial y$, and $\Delta\zeta$ we may state the following

THEOREM IV—Equation (6) *is equivalent to the following integrodifferential equation for the Fourier transforms:*

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) + 2\int_\Theta \frac{\theta_1\omega_2 - \theta_2\omega_1}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2, t)\overline{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2, t)d\theta$$

$$= -\nu(\omega_1^2 + \omega_2^2)Z(\omega_1, \omega_2, t) - \nu\Phi(\omega_1, \omega_2, t). \tag{50}$$

This equation, being fully equivalent to the Navier-Stokes equations, is the rational starting point for any rigorous study of the Fourier transform of the vorticity and hence, of the spectral function. This study seems to be very difficult; perhaps it will be only possible to check, in some approximate way, some of the working hypotheses which were assumed in recent papers, e.g., the assumption that the big eddies have a tendency to degenerate into smaller ones. This means that when t is increasing the peak on the spectral surface $\gamma(\omega_1, \omega_2, t)$ is gradually shifted towards the high frequencies ω_1, ω_2.

3. We can get interesting general results, if we consider the case where the flow extends over the entire plane.

$$X \qquad -\infty < x < +\infty, \qquad -\infty < y < +\infty.$$

This case may be regarded as a limiting case of previous problem. The boundary condition (3) means now that the fluid is at rest at infinity:

$$\lim_{r\to+\infty} u(x, y, t) = 0 \qquad \lim_{r\to+\infty} v(x, y, t) = 0. \tag{51}$$

$$r = +\sqrt{x^2 + y^2}.$$

If the kinetic energy is to remain finite,

$$E = \frac{1}{2}\int_X (u^2 + v^2)d\sigma < +\infty, \tag{52}$$

the decrease of u and v must be sufficiently rapid. A sufficient condition is

$$u = O(r^{-3/2}), \qquad v = O(r^{-3/2}) \quad \text{for large values of } r. \tag{53}$$

To insure the existence of the Fourier transform

$$Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_X \zeta(x, y, t)e^{-i(\omega_1 x+\omega_2 y)}d\sigma, \tag{54}$$

it is sufficient to assume that ζ is absolutely integrable

$$\int_X |\zeta| \, d\sigma < +\infty. \tag{55}$$

In this case, all the properties a) to j) enumerated for the Fourier transform (9) still hold for (54). The condition is obviously satisfied if:

$$\zeta = O(r^{-3}) \quad \text{for large values of } r. \tag{56}$$

To compute the Fourier transforms of $\partial\zeta/\partial t$, $u\partial\zeta/\partial x+v\partial\zeta/\partial y$, and $\Delta\zeta$, we can use the same process as before, taking for D the circle $x^2+y^2<R^2$ and then letting R tend towards $+\infty$. We must observe, however, that we no longer have $u=0$, $v=0$ on the boundary B; we must therefore carefully examine some terms in our equations.

In the evaluation of the Fourier transform of $u\partial\zeta/\partial x+v\partial\zeta/\partial y$, the two terms (44) and (45) do no longer vanish. According to Schwarz' inequality we have, however,

$$\left|\int_B \alpha u\zeta e^{-i(\omega_1 x+\omega_2 y)}ds\right| \leqq \int_B |u||\zeta| \, ds \leqq \left[\int_B u^2 ds\right]^{1/2}\left[\int_B \zeta^2 ds\right]^{1/2}.$$

From our assumption concerning the decrease of u and ζ, it is obvious that

$$\lim_{R\to+\infty}\int_B u^2 ds = 0, \qquad \lim_{R\to+\infty}\int_B \zeta^2 ds = 0.$$

Similar considerations apply to (45). Thus, at the limit, the expression (46) for the Fourier transform still holds.

In the evaluation of the Fourier transform of $\Delta\zeta$ we obtain a very fortunate simplification: from (48) ,we have

$$|\Phi| \leqq \frac{1}{4\pi^2}\int_B \left[\left|\frac{\partial\zeta}{\partial n}\right| + (|\omega_1| + |\omega_2|)|\zeta|\right]ds.$$

The assumption of decreasing of ζ yields

$$\lim_{R\to+\infty}\int_B \left|\frac{\partial\zeta}{\partial n}\right|ds = 0, \qquad \lim_{R\to+\infty}\int_B |\zeta| \, ds = 0;$$

thus

$$\lim_{R\to+\infty} \Phi = 0.$$

The Fourier transforms C and Z being extended to the whole plane, we have thus now:

$$C = -(\omega_1^2 + \omega_2^2)Z.$$

THEOREM V—In the case of a flow extending over the entire plane X, where the fluid is at rest at infinity, and where u, v, ζ decrease so rapidly that

$$E = \frac{1}{2}\int_X (u^2 + v^2)d\sigma < +\infty$$

and

$$\int_X |\zeta| \, d\sigma < +\infty,$$

the Fourier transform of the vorticity satisfies the integrodifferential equation:

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) + 2\int_\Theta \frac{\theta_1\omega_2 - \theta_2\omega_1}{\theta_1^2 + \theta_2^2} Z(\theta_1, \theta_2, t)\cdot \overline{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2, t)d\theta \tag{57}$$
$$= -\nu(\omega_1^2 + \omega_2^2)Z(\omega_1, \omega_2, t).$$

This equation seems to me the most simple that can be obtained in this manner.

To give at least an application, we shall establish the general solution of (57) for the particular, but interesting, class of flows, where the vorticity is constant along every stream line.

The flows characterized by $\zeta = F(\psi)$ have been the subject of interesting researches.[9] It is in this case and only in this case that non-linear terms disappear in the Navier-Stokes equations; their Fourier transform vanishes, of course. Equation (57) is then reduced to

$$\frac{\partial Z}{\partial t} = -\nu(\omega_1^2 + \omega_2^2)Z,$$

and the general solution is

$$Z(\omega_1, \omega_2, t) = Z_0(\omega_1, \omega_2)e^{-\nu(\omega_1^2+\omega_2^2)t}.$$

Here, $Z_0(\omega_1, \omega_2)$ denotes the arbitrary value of Z for $t = 0$. It is seen that, in this case, not only an upper bound of Z, but Z itself is decreasing exponentially for every value of ω_1, ω_2.

[9] Ratib Berker, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible*, Thesis, Lille, 1936.

THE METHOD OF THE HYPERCIRCLE IN ELASTICITY WHEN BODY FORCES ARE PRESENT*

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In an earlier paper[1] the method of the hypercircle in function space for boundary value problems in elasticity was developed under the restriction that body forces are absent. The purpose of the present note is to remove that restriction, and present the method in complete form, using the notation and basic ideas of PS.

Let an elastic body (in general anisotropic) be subjected to body forces X_i per unit volume, so that the equations of equilibrium are

$$E_{ij,j} + X_i = 0. \tag{1}$$

As regards boundary conditions, we shall suppose that the surface B of the body may be divided into regions designated by

$$[u_n, u_t], \qquad [u_n, T_t], \qquad [u_t, T_n], \qquad [T_n, T_t]. \tag{2}$$

Here $[u_n, u_t]$ indicates a portion of B on which both the normal component u_n of displacement and the tangential (vector) component u_t of displacement are assigned; $[u_n, T_t]$ indicates a portion of B on which the normal component of displacement and the tangential component of stress are assigned; and so on.

As in PS, a state of stress throughout the body corresponds to a vector in function space, and the scalar product is defined by $\mathbf{S}' \cdot \mathbf{S}'' = \int e'_{ij} E''_{ij} dv$, the integral being taken throughout the body. We set up the following notation:

$\mathbf{S}$ = natural state satisfying the equations of equilibrium and compatibility, and also the boundary conditions.

$\mathbf{S}'$ = basic stress state, satisfying the equations of equilibrium

$$E'_{ij,j} + X_i = 0, \tag{3}$$

and all boundary conditions on stress; this means that

$$\begin{aligned} T'_t &= T_t \quad \text{on} \quad [u_n, T_t], \\ T'_n &= T_n \quad \text{on} \quad [u_t, T_n], \\ T'_n &= T_n \quad \text{and} \quad T'_t = T_t \quad \text{on} \quad [T_n, T_t]. \end{aligned} \tag{4}$$

$\mathbf{I}'_p$ $(p = 1, 2, \cdots m)$ = orthonormal homogeneous stress states, each satisfying the equations of equilibrium *without body forces*,

$$E'_{(p)ij,j} = 0, \tag{5}$$

and making $T'_{(p)n}$ vanish wherever T_n is assigned and making $T'_{(p)t}$ vanish wherever T_t is assigned.

$\mathbf{S}''$ = basic displacement state; this corresponds to a state of stress which satisfies

* Received Aug. 25, 1947.

[1] W. Prager and J. L. Synge, Quarterly of Applied Mathematics, **5**, 241–271 (1947); this paper will be referred to as PS. *Cf.* also J. L. Synge, *The method of the hypercircle in function space for boundary value problems*, Proc. Roy. Soc. London (A) **191**, 447–467 (1947).

the equations of compatibility, so that a displacement u_j'' exists. This displacement satisfies all boundary conditions on displacement, so that

$$\begin{aligned} u_n'' &= u_n \quad \text{and} \quad u_t'' = u_t \quad \text{on} \quad [u_n, u_t], \\ u_n'' &= u_n \quad \text{on} \quad [u_n, T_t], \\ u_t'' &= u_t \quad \text{on} \quad [u_t, T_n]. \end{aligned} \tag{6}$$

$\mathbf{I}_q''$ $(q=1, 2, \cdots n)$ = orthonormal homogeneous displacement states, each corresponding to a state of stress satisfying the equations of compatibility, so that a displacement $u''_{(q)t}$ exists; this displacement is such that $u''_{(q)n}$ vanishes wherever u_n is assigned and $u''_{(q)t}$ vanishes wherever u_t is assigned.

It is important to note that, if k' and k'' are any constants, then $\mathbf{S}'+k'\mathbf{I}_p'$ belongs to the class of basic stress states $(\mathbf{S}')$ and $\mathbf{S}''+k''\mathbf{I}_q''$ belongs to the class of basic displacement states $(\mathbf{S}'')$.

We now proceed to prove that the extremity of $\mathbf{S}$ lies on a hypersphere determined by $\mathbf{S}'$ and $\mathbf{S}''$. By Green's theorem we have

$$\begin{aligned} \mathbf{S}'\cdot\mathbf{S}'' &= \int e_{ij}''E_{ij}'dv = \int u_{i,j}''E_{ij}'dv \\ &= \int u_i''T_i'dB - \int u_i''E_{ij,j}'dv, \end{aligned}$$

where dB is an element of the bounding surface B. Hence, by (3),

$$\mathbf{S}'\cdot\mathbf{S}'' = \int u_i''T_i'dB + \int u_i''X_idv. \tag{7}$$

But $\mathbf{S}$ belongs to the class of $\mathbf{S}'$ and also to the class of $\mathbf{S}''$. Thus we can substitute $\mathbf{S}$ for $\mathbf{S}'$ or $\mathbf{S}''$ in (7) and so obtain the following equations:

$$\begin{aligned} \mathbf{S}\cdot\mathbf{S}'' &= \int u_i''T_idB + \int u_i''X_idv, \\ \mathbf{S}'\cdot\mathbf{S} &= \int u_iT_i'dB + \int u_iX_idv, \\ \mathbf{S}\cdot\mathbf{S} &= \int u_iT_idB + \int u_iX_idv. \end{aligned} \tag{8}$$

By addition and subtraction of (7) and (8), we obtain

$$\mathbf{S}\cdot\mathbf{S} - \mathbf{S}\cdot\mathbf{S}' - \mathbf{S}\cdot\mathbf{S}'' + \mathbf{S}'\cdot\mathbf{S}'' = \int (u_i - u_i'')(T_i - T_i')dB. \tag{9}$$

For any pair of vectors, U_j, V_j, we have

$$\int U_iV_idB = \int U_nV_ndB + \int U_tV_tdB, \tag{10}$$

where U_n, V_n are normal components, and U_tV_t is the ordinary scalar product of the tangential vector components. We apply this reduction to the right hand side of (9),

and then split each integral into four parts corresponding to the four regions (2). It is easily seen from (4) and (6) that the result is zero, and so (9) reads

$$\mathbf{S}\cdot\mathbf{S} - \mathbf{S}\cdot\mathbf{S}' - \mathbf{S}\cdot\mathbf{S}'' + \mathbf{S}'\cdot\mathbf{S}'' = 0, \tag{11}$$

or, equivalently,

$$[\mathbf{S} - \tfrac{1}{2}(\mathbf{S}' + \mathbf{S}'')]^2 = \tfrac{1}{4}(\mathbf{S}' - \mathbf{S}'')^2. \tag{12}$$

This formula locates the extremity of $\mathbf{S}$ on a hypersphere in function space with center at

$$\mathbf{C}_0 = \tfrac{1}{2}(\mathbf{S}' + \mathbf{S}'') \tag{13}$$

and with radius R given by

$$R^2 = \tfrac{1}{4}(\mathbf{S}' - \mathbf{S}'')^2. \tag{14}$$

Since all that is required of $\mathbf{S}'$ and $\mathbf{S}''$ is that they belong respectively to the classes of basic stress states and basic displacement states, it follows from a remark made earlier that we may make the following substitutions in (11):

$$\text{for } \mathbf{S}' \text{ substitute } \mathbf{S}' + k'\mathbf{I}_p',$$
$$\text{for } \mathbf{S}'' \text{ substitute } \mathbf{S}'' + k''\mathbf{I}_q'',$$

where k', k'' are arbitrary constants. When we do this, and then subtract (11) from the result, we get an equation which must be satisfied for all values of k', k''. This yields

$$\left.\begin{aligned} \mathbf{S}\cdot\mathbf{I}_p' &= \mathbf{S}''\cdot\mathbf{I}_p' \qquad (p = 1, 2, \cdots, m), \\ \mathbf{S}\cdot\mathbf{I}_q'' &= \mathbf{S}'\cdot\mathbf{I}_q'' \qquad (q = 1, 2, \cdots, n), \end{aligned}\right\} \tag{15}$$

and also

$$\mathbf{I}_p'\cdot\mathbf{I}_q'' = 0 \qquad (p = 1, 2, \cdots, m;\ q = 1, 2, \cdots, n); \tag{16}$$

this last result shows that the set of homogeneous stress vectors is orthogonal to the set of homogeneous displacement vectors.

Equations (11) and (15) locate the extremity of the natural vector $\mathbf{S}$ on a hypercircle Γ. It is easy to prove that the center $\mathbf{C}$ of Γ and its radius R are given by

$$\mathbf{C} = \frac{1}{2}\left[\mathbf{S}' + \mathbf{S}'' - \sum_{p=1}^{m}\mathbf{I}_p'\{(\mathbf{S}' - \mathbf{S}'')\cdot\mathbf{I}_p'\} + \sum_{q=1}^{n}\mathbf{I}_q''\{(\mathbf{S}' - \mathbf{S}'')\cdot\mathbf{I}_q''\}\right], \tag{17}$$

$$R^2 = \frac{1}{4}\left[(\mathbf{S}' - \mathbf{S}'')^2 - \sum_{p=1}^{m}\{(\mathbf{S}' - \mathbf{S}'')\cdot\mathbf{I}_p'\}^2 - \sum_{q=1}^{n}\{(\mathbf{S}' - \mathbf{S}'')\cdot\mathbf{I}_q''\}^2\right]. \tag{18}$$

Let us use $\mathbf{I}_r$ $(r = 1, 2, \cdots, m+n)$ to denote the whole set of homogeneous vectors $\mathbf{I}_p'$, $\mathbf{I}_q''$, so that, by (16), $\mathbf{I}_r$ form a set of orthonormal vectors in function space. The equation of the hypercircle Γ may be written

$$\mathbf{S} = \mathbf{C} + R\mathbf{J}, \tag{19}$$

where $\mathbf{J}$ is arbitrary except for the equations

$$\mathbf{J}^2 = 1, \quad \mathbf{J}\cdot\mathbf{I}_r = 0 \qquad (r = 1, 2, \cdots, m + n). \tag{20}$$

We now seek the points on Γ at minimum and maximum distances from the origin O of function space. For any point $\mathbf{S}$ on Γ we have, by (19),

$$\mathbf{S}^2 = \mathbf{C}^2 + R^2 + 2R\mathbf{C}\cdot\mathbf{J}, \tag{21}$$

and so, on account of (20), for stationary values of $\mathbf{S}^2$ we have

$$\mathbf{C} = a\mathbf{J} + \sum_{r=1}^{m+n} b_r\mathbf{I}_r, \tag{22}$$

where a and b_r are undetermined multipliers. We deduce

$$b_r = \mathbf{C}\cdot\mathbf{I}_r, \tag{23}$$

and

$$a^2 = \Big[\mathbf{C} - \sum_{r=1}^{m+n} \mathbf{I}_r(\mathbf{C}\cdot\mathbf{I}_r)\Big]^2 = \mathbf{C}^2 - \sum_{r=1}^{m+n} (\mathbf{C}\cdot\mathbf{I}_r)^2. \tag{24}$$

For a stationary value of $\mathbf{S}^2$ we must take, by (22) and (23),

$$\mathbf{J} = a^{-1}\Big[\mathbf{C} - \sum_{r=1}^{m+n} \mathbf{I}_r(\mathbf{C}\cdot\mathbf{I}_r)\Big]. \tag{25}$$

We find it convenient to define a vector $\mathbf{G}$ by

$$\mathbf{G} = \frac{1}{2}\Big[\mathbf{S}' + \mathbf{S}'' - \sum_{r=1}^{m+n} \mathbf{I}_r\{(\mathbf{S}' + \mathbf{S}'')\cdot\mathbf{I}_r\}\Big], \tag{26}$$

so that

$$\mathbf{G}^2 = \frac{1}{4}\Big[(\mathbf{S}' + \mathbf{S}'')^2 - \sum_{r=1}^{m+n} \{(\mathbf{S}' + \mathbf{S}'')\cdot\mathbf{I}_r\}^2\Big]. \tag{27}$$

By (17) and (26), we have

$$\mathbf{C} = \mathbf{G} + \sum_{p=1}^{m} \mathbf{I}_p'(\mathbf{S}''\cdot\mathbf{I}_p') + \sum_{q=1}^{n} \mathbf{I}_q''(\mathbf{S}'\cdot\mathbf{I}_q''). \tag{28}$$

We note that $\mathbf{G}\cdot\mathbf{I}_r = 0$, and so by (28)

$$\mathbf{C}\cdot\mathbf{G} = \mathbf{G}^2 \geqq 0. \tag{29}$$

By direct algebra we may prove that

$$\mathbf{C} - \sum_{r=1}^{m+n} \mathbf{I}_r(\mathbf{C}\cdot\mathbf{I}_r) = \mathbf{G}. \tag{30}$$

Thus, by (24), $a = \pm|\mathbf{G}|$, and, by (25), $\mathbf{J} = \pm\mathbf{G}/|\mathbf{G}|$. By (19) the stationary values of $\mathbf{S}^2$, if $\mathbf{S}$ ranges over Γ, correspond to the vectors

$$\mathbf{C} \pm R\mathbf{G}/|\mathbf{G}|. \tag{31}$$

Of these two vectors, that with the + sign is the greater, on account of (29). Accordingly, we may say that the point of Γ farthest from the origin of function space has the position vector

$$\mathbf{V}' = \mathbf{C} + R\mathbf{G}/|\mathbf{G}|, \tag{32}$$

and the point nearest to the origin has the position vector

$$\mathbf{V}'' = \mathbf{C} - R\mathbf{G}/|\mathbf{G}|. \tag{33}$$

Thus we have the following inequalities for the magnitude of the vector $\mathbf{S}$ corresponding to the natural state of the elastic body:

$$\mathbf{V}''^2 \leqq \mathbf{S}^2 \leqq \mathbf{V}'^2, \tag{34}$$

or, by (29),

$$\mathbf{C}^2 + R^2 - 2R|\mathbf{G}| \leqq \mathbf{S}^2 \leqq \mathbf{C}^2 + R^2 + 2R|\mathbf{G}|, \tag{35}$$

where $\mathbf{C}$ is given by (17), R by (18), and $|\mathbf{G}|$ by (27), on taking the square root. We can show by direct algebra that

$$\mathbf{C}^2 + R^2 = \frac{1}{2}(\mathbf{S}'^2 + \mathbf{S}''^2) + \frac{1}{2}\Big[\sum_{p=1}^{m}\{(\mathbf{S}''\cdot\mathbf{I}_p')^2 - (\mathbf{S}'\cdot\mathbf{I}_p')^2\} + \sum_{q=1}^{n}\{(\mathbf{S}'\cdot\mathbf{I}_q'')^2 - (\mathbf{S}''\cdot\mathbf{I}_q'')^2\}\Big]. \tag{36}$$

The inequalities (34) or (35) bound the strain energy of the natural state below and above. It is interesting to note that the usual minimal principles bound the potential energy of the natural state, rather than its strain energy.

GENERAL IMPEDANCE—FUNCTION THEORY*

BY

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1. Introduction. In the field of lumped-constant circuits, considerable theory is available[1] concerning relations satisfied by all physically realizable impedances. Conditions which are both necessary and sufficient for physical realizability have been derived. It is the purpose of the present paper to extend some of these techniques to more general, distributed-constants circuits.

The basic method of existing theory consists in extending the impedance $Z = R + iX$ to "complex frequencies" $s = \gamma + i\omega$ so that Z becomes a complex function of a complex variable. In the field of lumped-constants, the circuit functions are rational and the extension is effected merely by substituting s for $i\omega$.

2. The concept of impedance. When we pass from the field of lumped-constant to more general circuits, we can no longer say that the impedances are rational functions of ω. This circumstance raises a number of difficulties. First, we must consider singularities other than poles, for it is known that all non-rational functions have at least one singularity which is not a pole. (For example, the input impedance of a shorted line, $\tanh(i\omega l/c)$, has an essential singularity at infinity.) Secondly, we no longer have the relatively simple type of differential equation which occurs in lumped-constant circuits. At our level of generality, there are available only Maxwell's Equations. These considerations make it advisable for us to review anew the general concept of impedance and the means of extending this concept to complex frequencies.

Before embarking on our general discussion, we must stress a point frequently overlooked in the foundations of electro-magnetic theory. When introducing this subject, the remark is often made that specialization to harmonic time dependence is of no great consequence due essentially to the Fourier Integral Theorem. At a later stage, it is then mentioned that the material constants in the resulting equations may depend upon frequency. Often it is *not* mentioned that, in the case of such dispersive media, it is obviously impossible to return to the original Maxwell's Equations for arbitrary time dependence.

Thus, if we wish to be accurate, we should not say merely that the postulates of electro-magnetic theory are contained in Maxwell's Equations. Rather, we must say, at least for dispersive media (and, in actual practice, in general), that the postulates consist of: (a) Maxwell's Equations for harmonic time dependence and (b) Fourier (or Laplace) superposition for arbitrary time dependence.

Our arguments will be based essentially upon (b). The purpose of the discussion was to emphasize that (b) is a basic *postulate* of electro-magnetic theory and is not a mere pseudo-mathematical consequence thereof.

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*Received May 7, 1947. The results and general method of this paper conform to a section of the author's doctor's thesis, the research for which was carried on under the direction of Dr. Ph. LeCorbeiller at Harvard University. The arguments, however, have been markedly changed from those originally employed.

[1]For an excellent summary of lumped-constant impedance-function theory, see H. W. Bode, *Network analysis and feedback amplifier design*, D. Van Nostrand, New York, 1945.

We must next decide what we shall mean when we speak of an impedance in general. We certainly shall want any impedance Z to satisfy a relation of the type:[2]

$$(\text{response quantity}) = (\text{applied quantity}) \times Z(i\omega). \tag{1}$$

Actually, we shall place no restrictions upon the quantities involved except for those limitations already implied by the form of (1). These implied restrictions, however, merit considerable discussion.

Firstly, note that applied and response quantities cannot always be interchanged. A familiar example in the field of lumped constants would be generator-voltage and short-circuit-output-current for a four terminal network. Under no conditions can the latter be considered "applied". (It is this impossibility of interchange which causes the analytic properties of a transfer function to differ markedly from those of its reciprocal.[1])

Secondly, it is specifically stated in (1) that Z is to depend only upon frequency. This implies that (*i*) the circuit configuration is constant in time and (*ii*) the circuit is linear (i.e. the material constants depend at most on frequency and spatial position). We shall add the further restriction that (*iii*) the circuit is passive; mathematically, this is the statement that the total power absorbed is eventually non-negative.

Finally, (1) implies that the applied and response quantities can be characterized by two single numbers, one of which is determined by the other. Thus the circuit must be so constructed that the entire spatial dependence of all field components is completely determined by frequency, while the absolute magnitude of the field at all points is determined by a single "excitation parameter".

We shall not attempt an exhaustive discussion of the cases in which this fourth restriction can be met, but a few remarks are in order. The first picture that comes to the reader's mind when this requirement is stated is probably a circuit which—while possibly containing waveguides, antennas, etc.,—has definite small input terminals. But what does "small" mean when the frequency has not been specified? Consider lumped-constant impedance-function theory. Here, definite input terminals are assumed and yet the impedance functions are treated as having physical significance beyond the frequency of visible light and, indeed, at $\omega = \infty$. What is essentially done, of course, is to idealize the physical picture in a self-consistent fashion so as to obtain a theory which is tractable and, at the same time, physically significant within the frequency range of interest. It seems reasonable that the same procedure could be carried out over greater frequency ranges; a case in point is transmission-line theory, which avoids the lumped-constant approximation of treating a line merely as a pair of leads but which ignores the effects of radiation, waveguide modes, and discontinuity reactances.

Following these thoughts one step further, we note that we may dispense with the concept of definite input terminals. We might, for example, choose as the quantities in (1), the transverse dominant-mode components of the forwardly-propagated electric and magnetic fields at some point of a reference plane located within an input waveguide. The characteristics of the waveguide would assure satisfaction of the above condition provided that the higher modes could be excited only through the dominant mode. This could be accomplished by specifying that the guide be fed by an input guide which "tapers to a point generator". In practice this last phrase could mean that

[2]Note that the usual roles of applied and response quantities have been interchanged. This facilitates treatment of transfer impedances.

the taper is to be cut off and attached to an actual generator after the cross-section has become too small to support higher modes; this would involve a set of generators with associated coupling tapers of different lengths for each frequency range (no taper is needed below the cut-off of the main guide), but the impedance as defined above would, nevertheless, be well-defined.

Having reached this level of generality, we are faced with the question of distinguishing between transfer and driving-point impedances. We shall adopt the following definition, the motivation for which is obvious in the light of the wave-guide example.

Definition: If the product of the quantities in (1) has the same algebraic sign as the total power (or energy) absorbed by the circuit, then Z is called a driving-point impedance.

Before closing this section, another consideration should be mentioned. Inasmuch as our arguments will be based upon physical principles, we must require that the circuit can actually be constructed. The significance of this remark will, we hope, become clearer as the argument progresses. For the present, we may remark that a transmission line or waveguide of infinite length must therefore be excluded; though Maxwell's Equations are capable of treating this fiction, the device cannot actually be exhibited in the world of reality. An antenna radiating into empty space (or into a completely absorbing box if we wish to avoid the non-linear ionosphere) can be constructed and used and is therefore *not* to be excluded. Secondly, we should note that truly lossless circuits are also a fiction. We do not wish to exclude these; they are far too useful. However, recognizing their idealized nature, we shall, when convenient, treat them as the limiting case of a lossy circuit. This corresponds to the physical facts: losses are present but, in some cases, effects due specifically to dissipation are small enough to be neglected.

3. Necessary conditions. We now wish to extend our impedance $Z(i\omega)$ to complex frequencies $s = \gamma + i\omega$. Maxwell's Equations certainly have formally valid mathematical solutions for time dependence of the form $e^{\gamma t}\cos(\omega t) = Re(e^{st})$. These solutions can be obtained by mere substitution. With dispersive media, however, there would be considerable doubt as to the correct value of the material constants at such complex frequencies. Moreover, although the substitution process will usually lead to an analytic extension of the original function of a single variable, there exist functions (e.g. jump functions) which cannot be so extended. We shall, therefore, adopt a very different approach, which allows the use of the powerful theory of the Laplace Transformation.

Let us cause the applied quantity, $f(t)$ say, to vary with time in an arbitrary manner, subject only to the condition $f(t) = 0$ for $t < 0$ (without which we could hardly be said to "apply" the quantity). Let $g(t)$ represent the response quantity; since the circuit cannot respond before it is stimulated, $g(t) = 0$ for $t < 0$. If, for the moment, $s = i\omega$, then, by the fundamental superposition *postulate* of electro-magnetic theory ((b) of Section II), we have:

$$Z(s) = \frac{\int_0^\infty e^{-st} g(t)\, dt}{\int_0^\infty e^{-st} f(t)\, dt}. \tag{2}$$

If we now allow s to take on any complex value, the integrals in (2) become the definition of the Laplace Transformations of f and g. Thus, by analytic continuation if neces-

sary, $Z(s)$ has been defined for all complex s. The superposition postulate of electromagnetic theory implies, in particular, that $Z(s)$ as defined by (2) takes the same values on $s = i\omega$ whatever the function $f(t)$ may be. We shall see later that Z is analytic in $\gamma > 0$[3] and, at least, on portions of $\gamma = 0$. By an identity theorem[4] for analytic functions, it then follows that $Z(s)$ is well-defined by (2) despite the arbitrariness of $f(t)$. We shall also see later that Z has the usual physical significance in $\gamma > 0$.

The definition (2) has the great advantage that we may, in particular, choose harmonic time variation of extremely low frequency and thereby utilize established facts.

In accordance with the last remark of Section II, we shall first consider circuits of arbitrarily minute but nevertheless finite loss. Moreover, let $f(t)$ vary harmonically in time at such a frequency $s_0 = i\omega_0$ that the associated wavelength is far greater than the overall ("metallic") dimensions of the circuit. By this statement, we do *not* mean that circuit radiation is to be neglected in the entire analysis but merely that the above wavelength is to be so chosen that radiation at *that* frequency will be extremely small. (For example, a centimeter antenna may be considered as a capacitance at a frequency of one cycle/sec.) Thus, though the circuit may in a sense be infinite, the wavelength is to be chosen very large compared to its material, *physical* dimensions.[5]

Under these conditions ($f(t) = e^{s_0 t}$, $t > 0$, s_0 imag.),[6] the (presumably) established theories and facts of low-frequency electricity show that the response quantity will be given by

$$g(t) = \begin{cases} 0 & t < 0, \\ Ae^{s_0 t} + h(t) & t > 0, \end{cases} \tag{3}$$

where $h(t)$ represents the transient response and is bounded by e^{-at} with small but finite a.[5] (Recall our momentary restriction of finite losses.)

Carrying out the Laplace Transformations in (2), we have:

$$Z(s) = A + (s - s_0)H(s), \tag{4}$$

where $H(s)$ is the Laplace Transform of $h(t)$. Since $h(t)$ is bounded by e^{-at}, it follows[7] that $H(s)$ is analytic in the region $Re(s) \equiv \gamma > -a$. Thus we may finally state the following theorem.

For a circuit with finite loss, however small, there exists an $a > 0$ such that $Z(s)$ is analytic in $\gamma > -a$.

Setting $s = s_0$, (4) immediately gives $Z(s_0) = A$, the correct value according to (3). If we choose another value for s_0, the transient excitation may change somewhat (as it must by (4) since Z is well-defined), but its general character cannot be greatly altered since transient behavior is essentially independent of frequency. (A physicist accustomed to working at high frequencies—and thus with short times—may, however, insist on

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[3]Throughout this paper the phrase "in $\gamma > 0$", in accordance with the usual convention, does *not* assert anything about the point $s = \infty$. As mentioned earlier, an essential singularity often occurs there.

[4]C. Carathéodory, *Conformal representation*, Cambridge Univ. Press, 1941 (reprinted), p. 82.

[5]Note that it is precisely at this point that the fictitious infinite transmission line violates the restriction of "physical reality" laid down in Section 2.

[6]For simplicity, we use complex notation. The reader can easily verify explicitly that the same final results are obtained for the true time variations.

[7]G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Dover Publications, New York, 1943, pp. 43, 194, 401.

preserving a "fine-structure" which the audio worker chooses to neglect. For example, compare the response to a unit step of a short lossless line terminated by a resistor with that computed by assuming the line to be a small inductance—or even a mere pair of leads.) Thus $H(s)$ will remain analytic in $\gamma \geq 0$ and therefore bounded at any point in this region. Consequently $Z(s_0)$ will reduce to the correct value A whatever complex value of s_0 is assumed. The usual physical significance may therefore be attached to $Z(s)$ for complex values of s lying in $\gamma \geq 0$.

We also note that if s is real in (2) then $Z(s)$ is real and thus, by the Schwarz reflection principle,

$$Z(s^*) = Z^*(s). \tag{5}$$

This also follows formally from the reality of the general form of Maxwell's Equations.

We must, finally, consider circuits which, under appropriate conditions and perhaps only within certain frequency ranges, may be considered "lossless". Inasmuch as such circuits, for reasons given earlier, should really be considered as the limiting case of a lossy circuit, we may suspect that the theorem italicized above will "essentially" hold in $\gamma \geq 0$, and this will prove to be true.

At sufficiently high frequencies, a dissipationless circuit must be a lossless cavity. For lower frequencies we may arrive at the same situation in the following way. It may be that the circuit is not truly a cavity but produces practically zero radiation. Since this very small radiation is neglected anyway, we may surround the circuit by a sufficiently large (but finite) perfectly conducting box; the reflected radiation will then be of the same order of magnitude as that originally neglected.

Now we wish to re-examine the argument leading to (4) and the theorem following therefrom. Certainly we may still choose an s_0 whose wavelength is very much larger than the circuit and its surrounding box. The transient response $h(t)$ in (3) is, again by the established facts of low-frequency systems, bounded but never dies away. The transient will consist of a superposition of the resonant frequencies of the circuit. Now it is shown in the theory of eigen-value problems[8] that, for a lossless cavity, these resonances occur at discrete, isolated frequencies (i.e. they form a sequence with no finite limit point). Thus we may say[6]

$$h(t) = \sum a_n e^{i\omega_n t}. \tag{6}$$

(An added exponentially decaying part may occur if the circuit is not "lossless" at all frequencies; this, it is easily seen, contributes no new information about the region $\gamma \geq 0$.) The Laplace Transform of (6) is

$$H(s) = \sum a_n/(s - i\omega_n). \tag{7}$$

Substituting (7) into (4), we see that $Z(s)$ is analytic in $\gamma > 0$ (as follows merely from the boundedness of $h(t)$),[7] and, since the ω_n have no finite limit point, we may summarize our results as follows. These are valid for both transfer and driving-point impedances.

(*a*) $Z(s)$ *is analytic in* $\gamma \geq 0$ *except only for isolated poles on* $\gamma = 0$;

(*b*) $Z(s^*) = Z^*(s)$.

Finally, we wish to find an added condition to characterize driving-point impedances

[8] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Interscience Publishers, New York, 1943, vol. 1, pp. 358, 384.

as defined in Sec. 2. This has been done by Brune[9] who has shown that if the integrated product $f(t)g(t)$ is always positive for all sufficiently large time intervals then:

$$Re(Z) = R \geq 0 \qquad \text{in the region} \qquad \gamma \geq 0 \tag{8}$$

This argument establishes (8) for lossy circuits, and treatment of a "lossless" circuit as a limit shows that (8) is valid generally.

4. Consequences of necessary conditions. Throughout the remainder of this paper, only driving-point impedances will be considered. For brevity, we shall refer to such merely as "impedances".

Any physical impedance must satisfy any theorems derived from the conditions italicized above and Eq. (8). The author has made a study of the class of functions defined by these conditions; this investigation is purely mathematical and has appeared elsewhere.[10] The results of physical interest will be summarized in this section.

The only impedance which is purely resistive at all (real) frequencies is the constant resistance $Z \equiv R_0$.[11]

The purely reactive functions have many properties analogous to those given for lumped-constant reactances in the well-known Foster's Reactance Theorem. General reactances are analytic except on the real frequency axis, and there the zeros and poles alternate, while the origin is always a zero or pole. The derivative is always positive (never zero), and for all ω,

$$dX/d\omega \geq | X/\omega | \tag{9}$$

Previously, (9) had been obtained only from Lagrange's equations for a lumped-constant network; it is now known to hold for all generalized reactances.

A general reactance is necessarily a complicated function, for, if it has only a finite number of poles on the real frequency axis, it must be a rational function and therefore a lumped-constant reactance. Nevertheless, two expansions used for lumped-constant reactances can be extended to general reactances; the processes involved naturally become infinite. All of the expansions quoted below converge absolutely and uniformly in any bounded region. (Except, of course, for terms involving poles within the region.)

One such expression is the "partial fraction" expansion:

$$Z = As + (K_0/s) + \sum_1^\infty 2K_n s/(s^2 + \omega_n^2) \tag{10}$$

where ω_n are the anti-resonant (angular) frequencies and K_n, the residues at these poles (K_0 may be zero); $A = \lim Z/s$ as s approaches ∞ in $| \arg s | < \pi/2$.

Schelkunoff has suggested[12] that there might exist such expansions and, pending their proof, has shown in considerable detail how they may be used to calculate the input reactance of complicated distributed-constant structures. The anti-resonant

[9] O. Brune, *Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency*, J. Math. Phys. **10**, 191-236 (1931).

[10] P. Richards, *A special class of functions with positive real part in a half-plane*, Duke Math. Journal, **14**, 777–786 (1947).

[11] Proved for lumped-constant circuits by O. Zobel, *Distortion correction in electrical circuits with constant resistance recurrent networks*, Bell Syst. Tech. J. **7**, 438-534 (1928).

[12] S. A. Schelkunoff, *Representation of impedance functions in terms of resonant frequencies*, Proc. I. R. E., vol. 32, pp. 83-90 (Feb. 1944).

frequencies are, of course, relatively easily evaluated. The K_n (i.e. equivalent L and C) can be found from energy storage. The first term is less easily calculated; it can be shown to be $(d/ds)[Z - (K_0/s)]$ at $s = 0$ minus the series $\sum_1^\infty 2K_n/\omega_n^2$. For this reason and also because the series (10) converges rather slowly, it is preferable to employ the following. Use Z or $Y = 1/Z$, whichever is zero at zero frequency. Then, setting $Z'(0)$—or $Y'(0)$—equal to L (the low-frequency inductance—or capacitance), we have

$$Z = Ls + 2s^3 \sum_1^\infty K_n/[\omega_n^2(s^2 + \omega_n^2)] \tag{11}$$

This series converges much faster than (10) and the terms are more easily evaluated. For examples of such calculations, the reader is referred to Schelkunoff, op. cit.

The validity of (10) justifies the common practice of approximating a resonant cavity with an equivalent lumped circuit in the vicinity of resonance.

Also, if, in any (or all) terms of (10) or (11), s is replaced by $s + \delta(\delta > 0)$, the series converges as before but now represents a circuit with small losses; K_n and ω_n may, to a first approximation, be evaluated as above, and the value of δ for each term can then be obtained from the "Q" of the circuit at that resonant frequency.

Another useful expansion is analogous to Foster's cannonical form for lumped-constant reactances. Here it becomes an infinite product. If $Z(0) = 0$,

$$Z = Ls \prod_1^\infty \frac{1 + (s^2/\Omega_n^2)}{1 + s^2/\omega_n^2} \tag{12}$$

while if $Z(0) = \infty$,

$$Z = \frac{1}{Cs} \prod_1^\infty \frac{1 + (s^2/\Omega_n^2)}{1 + (s^2/\omega_n^2)} \tag{13}$$

where L or C is the low-frequency inductance or capacitance, Ω_n are the series-resonant (angular) frequencies, and ω_n the anti-resonant frequencies (both numbered by increasing magnitude).[13]

The various terms in (12, 13) are much more easily evaluated than those in (11). This is somewhat compensated by slower convergence. As a simple example of the use of these products in calculation of reactances, consider a shorted line. Suppose we had found (say, merely by considering traveling-wave reflection) that the line is series-resonant at $\Omega_n = n\pi c/l$ and anti-resonant at $\omega_n = (2n + 1)\pi c/2l$. The low-frequency inductance is then easily calculated, and substitution in (12) immediately gives the reactance at all frequencies. In this simple case, the product can be evaluated for general s because it is known to be the infinite product expansion of $L \tanh (ls/c)$. These same principles can obviously be extended to any circuit and will yield impedance formulas not easily obtained in any other fashion.

We turn now to general, non-reactive impedances. The resistance will, in general, assume an *absolute* minimum at some point because $R \geq 0$. At such a point, the impedance behaves locally very much like a reactance. The derivative $dZ/ds = dX/d\omega$ is real and positive and even satisfies (9). The lower bound R_0 of R represents essentially

[13] After developing (12, 13), the author discovered the recently available article, K. Franz, *Das Reactanztheorem für beliebige Hohlräume*, Elek. Nach. Tech., Bd 21, Heft 1/2, S 8, (Jan/Feb 1944).

Franz proves (12, 13) for cavities of arbitrary shape. For a comparison of his work and that of the present author, see Appendix B.

a constant series resistance because the function $Z - R_0$ again satisfies (8) and the italicized conditions immediately above.

Except for an added pure reactance, the resistive part of a generalized impedance specifies its reactance completely. If $R(\omega)$ is the resistive part (on $\gamma = 0$), the formula is:

$$X(c) = \frac{2c}{\pi} \int_0^\infty \frac{R(t) - R(c)}{t^2 - c^2}\, dt \tag{14}$$

This formula was proved for the case of lumped-constant circuits by Bode.[1]

The various other formulas of Bode have not yet been extended for the completely general case. The original proofs were based on Cauchy's Integral Formula and require that various integrals over a large semi-circle in the right half-plane approach zero with increasing radius. For rational functions, this proof offers no great difficulty, but a general impedance will have an essential singularity at infinity. This means that the behavior of Z, as s approaches infinity, may vary radically with the direction of approach.

If, however, Z is bounded on the real frequency axis, then it may be shown that it is bounded in the entire right half-plane. This fact immediately makes many[14] of Bode's formulas available for this class of functions.

However, the phase-area law:

$$\int_{f=0}^{f=\infty} X\, d(\log f) = \frac{\pi}{2}(R(\infty) - R(0))$$

is certainly not valid unless $R(\infty)$ has meaning. Even if Z is bounded, $R(\infty)$ may not exist; for example, the impedance of a shorted section of slightly lossy line, $\tanh(b + s)$ $(b > 0)$, is bounded on the real frequency axis, but oscillates indefinitely as $\omega \to \infty$. If Z is bounded *and* $R(\infty)$ exists, then Z will approach $R(\infty)$ uniformly in the entire right half-plane. The phase-area law is then easily proved by Bode's method[15] for this class of functions.

Appendix A

It is well-known that the characteristic impedances of transmission lines and waveguides may have branch points on the real-frequency axis. We have indicated in Sec. 2 our reasons for excluding infinite lines or guides, but it might be well to point out explicitly why the input impedance of a finite line or guide cannot have such a branch point if the terminating impedance has none.

Recall that the propagation function θ of a line or guide has exactly the same branch points as the characteristic impedance Z_0. In both cases, the point is such that varying the complex frequency once around the branch point merely changes the sign of Z_0 and θ.

Now, in the expression for the input impedance, Z_0 and θ appear only in the combinations $Z_0 \tanh \theta L$ and $(\tanh \theta L)/Z_0$. Since the hyperbolic tangent is an odd function of its complex variable, these expressions do not change at all when the point in question is circled. Therefore, by definition, the input impedance does not have a branch point.

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[14]Specifically (Bode, op. cit.):
p 301, No's I(b), I(d), III(b), IV(b), V(c), VI(b)
p 335, No's I, III, IV, V

[15]Bode, op. cit., p 286. Use $(Z - R(\infty))/s$ as the integrand instead of Z/s. Formulas II, p 335, are also valid for this class of functions.

Appendix B

It was mentioned in Sec. 4 that Franz has proved (12), (13). His method employs the theory[8] of the distribution of the eigen-values of the wave equation. When the developments of this paper were originally evolved, the author was unaware of these theories and so did not use the following condition, which is an immediate corollary of the present argument of Sec. 3.

$\mathrm{Lim}(n(f)/f)$ *as* f *approaches* ∞ *exists, where* $n(f)$ *is the number of resonances below* f.

Franz uses this condition in his proof. Since it may be shown that (8) and the italicized conditions immediately above (8) do not imply this condition, the author's theorem is, from a purely mathematical point of view, more general than that of Franz.

Similarly, the addition of this extra condition does not invalidate any of the present results since it merely means working with a subclass of those functions satisfying (8) and the italicized conditions above it. It is quite likely, however, that restrictions such as that just given could be used to obtain further results not implied by our earlier conditions alone. Consideration of slightly lossy cavities immediately suggests that a more general restriction may exist on the asymptotic behavior of the number of maxima of R on the real frequency axis.

A GENERAL APPROXIMATION METHOD IN THE THEORY OF PLATES OF SMALL DEFLECTION*

BY

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Introduction. The methods developed up to now in the theory of plates do not lend themselves readily to the solution of the problem in the case of an arbitrary shape of the plate. The papers of Nadai, Navier, Reissner, Southwell, and others, deal only with plates of special shapes. In the present paper the author develops a new method which aims at the solution of the problem quite independently of the shape of the plate. The outstanding feature of this method is that it is entirely independent of the shape of the plate; its success depends, however, on the type of loading. If there is no lateral load, i. e., if the forces act in the plane of the plate, the method is successful independently of whether the forces are distributed uniformly or not. Thus, the critical stresses of a plate of arbitrary shape can be found. If there is a lateral load, the method is not always successful. There are cases in which it is not possible to find a particular solution or a fundamental integral of a non-homogeneous linear partial differential equation of the fourth order with the aid of the methods now available in mathematics.

In general, the method involves the following two steps:

(a) to obtain a function which satisfies the linear partial differential equation of the fourth order for the deflection of the plate, but does not satisfy the boundary conditions,

(b) to obtain a solution of this partial differential equation which satisfies the boundary conditions precisely or approximately.

In the case of a linear homogeneous partial differential equation of the fourth order with constant coefficients, there exists the possibility of separating the variables, and a complete set of functions can generally be found with the aid of conventional methods. The solution then is a simple series of these functions with coefficients which are determined from the boundary conditions. In the case of a linear homogeneous partial differential equation of the fourth order with variable coefficients, the problem becomes much more complicated since, in this case, the variables cannot be separated. A method given by S. Bergman [1]† then leads to a complete set of functions, and in particular to convergent expansions in terms of these functions. The complete solution of the differential equation is given by a double series.

To keep the possibility of applying the present method to the case where the forces acting in the plane of the plate are not distributed uniformly, the author chose Bergman's method. In order not to obscure the whole procedure by long calculations, the forces have been assumed to be uniformly distributed in the plane of the plate in the numerical example elaborated in the present paper; i.e., the coefficients in the linear homogeneous partial differential equation of the fourth order are assumed constant.

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†Numbers in the brackets refer to the bibliography at the end of the paper.

However, this example illustrates the general method. In the case where the forces in the plane of the plate are not distributed uniformly, the general method remains unchanged. On the other hand, some changes have to be introduced into the numerical procedure and this has been explained throughly in Ref. 10. Thus, with a certain amount of additional numerical work, the present method may be extended to the general case of variable coefficients. The boundary conditions are only satisfied approximately with the present method.

A brief resume of Bergman's method of solving homogeneous linear partial differential equations of the fourth order is given in Sec. 1. Since this method represents the solution in the form of infinite series, which involve in practical problems many thousands of operations of a simple nature, this technique is particularly suitable for punch-card machines. Such machines were employed in obtaining the numerical results given in this paper. The functions appearing in the infinite series are defined and evaluated, and the various partial differential equations of loaded plates with small deflections are discussed in Sections 2 and 3. As an illustration of the procedure, the method has been applied in Sec. 4 to the calculation of critical stresses in a triangular plate loaded uniformly in its plane parallel to the x-axis. The example shows that once the scheme for the calculations has been set up, the computations are quite straightforward and simple and may be performed by an inexperienced staff. All details of the example are explained together with the method of avoiding solution of determinants of higher order. Several final remarks close the paper.

1. Solution of linear partial differential equations of the fourth order. The proof and complete explanation of this particular method of solving linear partial differential equations of the fourth order can be found in S. Bergman's paper [1]. Only a brief resume of this method will be given here. The differential equation considered is the following:

$$U_{zzz^*z^*} + MU_{zz} + LU_{zz^*} + NU_{z^*z^*} + AU_z + DU_{z^*} + CU = 0. \tag{1}$$

The coefficients, M, L, $\cdots$ C are assumed to be entire functions of z and z^* where, in terms of complex notation, $x + iy = z$, $x - iy = z^*$. The function U is also assumed to be a function of z and z^*. When z and z^* are conjugate complex numbers, x and y are restricted to real values. If x and y are allowed to take on complex values, then z and z^* will be independent complex variables. The asterisk denotes the conjugate complex number and partial derivatives are denoted by subscripts. If one sets $U = u + iv$, Eq. (1) is equivalent to a system of two real fourth order partial differential equations for two real functions u and v. If for real values of x and y the functions L and C are real and $M = N^* = M_1 + iM_2$, and $A = D^* = A_1 + iA_2$, then the first of these two equations contains u only and has the form:

$$\Delta\Delta u + au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0, \tag{2}$$

where

$$\Delta\Delta u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^2},$$

$$a = 4L + 8M_1, \qquad b = 8M_2, \qquad c = 4L - 8M_1, \tag{3}$$

$$d = 16A_1, \qquad e = 16A_2, \qquad f = 16C.$$

In applications, we are mostly interested in real equations of the form (2). However, it is simpler to work with the general equation (1) without making any special assumptions about the coefficients. The derivatives employed are the following:

$$U = u + iv,$$

$$U_z = \frac{\partial U}{\partial z} = \frac{1}{2}\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right), \qquad U_{z^*} = \frac{\partial U}{\partial z^*} = \frac{1}{2}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right),$$

$$U_{zz^*} = \frac{\partial}{\partial z^*}\left[\frac{1}{2}\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right)\right] = \frac{1}{4}\left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right) + i\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \tag{4}$$

$$U_{zz} = \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} - 2i\frac{\partial^2 U}{\partial x\,\partial y} - \frac{\partial^2 U}{\partial y^2}\right),$$

$$U_{z^*z^*} = \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + 2i\frac{\partial^2 U}{\partial x\,\partial y} - \frac{\partial^2 U}{\partial y^2}\right),$$

$$U_{zzz^*z^*} = \frac{1}{16}\left(\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + \frac{\partial^4 U}{\partial y^4}\right).$$

S. Bergman [1] has proved the following theorem concerning the solution of Eq. (1) [see Theorem 2.1. and Remark 9.1. of his paper].

Theorem. There exist four functions $E^{(k,\kappa)}(z, z^*, t)$, $k = I, II$, $\kappa = 1, 2$, which are defined for sufficiently small values, say $|z| < \rho_1$, $|z^*| < \rho_2$, and for $|t| \leq 1$, possessing the following property: If $f_\kappa(\zeta)$ and $g_\kappa(\zeta)$, $\kappa = 1, 2$, are any analytic functions of ζ defined in the neighborhood of the origin, then

$$U(z, z^*) = \sum_{\kappa=1}^{2}\int_{-1}^{+1}\left\{E^{(I\kappa)}(z, z^*, t)f_\kappa\left[\frac{1}{2}z(1 - t^2)\right] + E^{(II\kappa)}(z, z^*, t)g_\kappa\left[\frac{1}{2}z^*(1 - t^2)\right]\right\}(1 - t^2)^{-1/2}\,dt, \tag{5}$$

is a solution of (1). Conversely, if $U(z, z^*)$ is a solution of (1) defined in the neighborhood of the origin $z = z^* = 0$, then U can be represented in the form (5) by means of suitably chosen functions f_κ and g_κ, $\kappa = 1, 2$.

This theorem yields a local result; i.e., this theorem states that representation (5) for $U(z, z^*)$ is valid only in a sufficiently small neighborhood of the origin. By proving a theorem concerning analytical continuation in the complex domain of solutions of the differential equation (1) with analytic coefficients, Bergman has shown that representation (5) is valid in the large i.e. in a finite domain. He also proved the following theorem [Theorem 11.1].

Theorem. Let U be a solution of (1) represented in the form (5). Let U be regular for $(x, y)\varepsilon R^2$, $E^{(k,\kappa)}(z, z^*, t)$ for $|t| \leq 1$, $(x, y)\varepsilon R^{*2} + \frac{1}{2}R^{*2}$. Then the functions f_1, f_2, g_1, g_2 are regular in $\frac{1}{2}R^2$. By $\frac{1}{2}R^2$ we denote the domain obtained from domain R^2 by the transformation $z^{\blacktriangle} = \frac{1}{2}z$.

Because of the condition $|t| \leq 1$ in (5) this domain cannot be greater than the unit circle. In order to obtain rapid convergence of the series, the radius of the circle should not be greater than 0.5 to 0.6 for practical purposes. The functions $E^{(k,\varkappa)}(z, z^*, t)$ introduced above have the property that

$$\begin{aligned} E^{(I1)}(z, 0, t) &= E^{(II1)}(0, z^*, t) = 1, \\ E^{(I2)}(z, 0, t) &= E^{(II2)}(0, z^*, t) = 0, \\ E_{z^*}^{(I1)}(z, 0, t) &= E_z^{(II1)}(0, z^*, t) = 0, \\ E_{z^*}^{(I2)}(z, 0, t) &= E_z^{(II2)}(0, z^*, t) = 1. \end{aligned} \tag{6}$$

Each of the functions $E(z, z^*, t)$ can be represented in the form

$$E(z, z^*, t) = P^{(0)}(z, z^*) + \sum_{\nu=1}^{\infty} t^{2\nu} z^{\nu} P^{(\nu)}(z, z^*), \tag{7}$$

where the functions $P^{(\nu)}$ are to be determined. They are defined by the following system of differential equations, which arise from the application of the linear differential operator $L_1^{\blacktriangle}(E)$ and from the solution of the equation $L_1^{\blacktriangle}(E) = 0$:

$$\begin{aligned} D_1[P^{(0)}] &= 0, \\ D_1[P^{(1)}] &= -4D_1[P_z^{(0)}] - 2D_2[P^{(0)}], \\ D_1[P^{(n+2)}] &= -\frac{1}{n^2 + 2n + 3/4}\,[D_1[P_{zz}^{(n)}] + (2n + 1)D_1[P_z^{(n+1)}] \\ &\quad + D_2[P_z^{(n)}] + \left(n + \frac{1}{2}\right)D_2[P^{(n+1)}] \\ &\quad + NP_{z^*z^*}^{(n)} + DP_{z^*}^{(n)} + CP^{(n)}], \qquad (n = 0, 1, 2 \cdots), \end{aligned} \tag{8}$$

where

$$D_1(H) = H_{z^*z^*} + MH \text{ and } D_2(H) = LH_{z^*} + AH. \tag{9}$$

System (8) is a system of differential recurrence formulas for the functions $P^{(n)}$. After $P^{(0)}$, $P^{(1)}$, $\cdots$ $P^{(n+1)}$ have been determined, $P^{(n+2)}$ can be obtained by solving the ordinary differential equation, where only $P^{(n)}$ and $P^{(n+1)}$ appear. There exist two sequences of functions $P^{(I1n)}(z, z^*)$ and $P^{(I2n)}(z, z^*)$, $n = 0, 1, 2 \cdots$, satisfying the differential equations (8), $n = 0, 1, 2 \cdots$, and such that

$$\begin{aligned} &P^{(I10)}(z, 0) = 1, \qquad P_{z^*}^{(I10)}(z, 0) = 0, \\ &P^{(I1n)}(z, 0) = 0, \qquad P_{z^*}^{(I1n)}(z, 0) = 0, \qquad (n = 1, 2 \cdots), \end{aligned} \tag{10}$$

and

$$\begin{aligned} &P^{(I20)}(z, 0) = 0, \qquad P_{z^*}^{(I20)}(z, 0) = 1, \\ &P^{(I2n)}(z, 0) = 0, \qquad P_{z^*}^{(I2n)}(z, 0) = 0, \qquad (n = 1, 2 \cdots). \end{aligned}$$

The conditions and the proof of uniform and absolute convergence of these series have been given by Bergman [1]. Two other series for $P^{(\nu)}$ are obtained by interchanging z and z^*, by interchanging differentiation with respect to z and z^* everywhere and by replacing M and A by N and D respectively, and conversely.

Let us take into consideration only one function E. Substituting Eq. (7) into (5), one obtains:

$$U(z, z^*) = \int_{-1}^{+1} P^{(0)}(z, z^*) f\left[\frac{z(1-t^2)}{2}\right](1-t^2)^{-1/2}\,dt + \int_{-1}^{+1} t^2 z P^{(1)}(z, z^*) f\left[\frac{z(1-t^2)}{2}\right](1-t^2)^{-1/2}\,dt + \int_{-1}^{+1} t^4 z^2 P^{(2)}(z, z^*) f\left[\frac{z(1-t^2)}{2}\right](1-t^2)^{-1/2}\,dt + \cdots \tag{11}$$

Let us assume that the integral

$$I_1 = \int_{-1}^{+1} f\left[\frac{z(1-t^2)}{2}\right](1-t^2)^{-1/2}\,dt,$$

may be expanded into a simple power series of the form

$$I_1 = \int_{-1}^{+1} f\left[\frac{z(1-t^2)}{2}\right](1-t^2)^{-1/2}\,dt = \sum_{\nu=0}^{\infty} z^\nu. \tag{11a}$$

To this end let us assume that the function $f(z, t)$ under the integral sign can be expanded into a power series of the form

$$f\left[\frac{z(1-t^2)}{2}\right] = \sum_{\nu=0}^{\infty} c_\nu \frac{z^\nu (1-t^2)^\nu}{2^\nu}, \tag{11b}$$

This series (11b) can then be inserted in Eq. (11a), giving:

$$I_1 = \int_{-1}^{+1} f\left[\frac{z(1-t^2)}{2}\right](1-t^2)^{-1/2}\,dt = \int_{-1}^{+1} \sum_{\nu=0}^{\infty} c_\nu \left[\frac{z^\nu (1-t^2)^\nu}{2^\nu}\right](1-t^2)^{-1/2}\,dt \tag{12}$$

$$= \sum_{\nu=0}^{\infty} z^\nu.$$

If we make the change of variable $t^2 = p$, then

$$dt = \frac{dp}{2(p)^{1/2}}, \tag{13}$$

and

$$\int_{-1}^{+1} (1-t^2)^{\nu-1/2}\,dt = \int_0^1 (1-p)^{\nu-1/2} p^{-1/2}\,dp = \frac{\Gamma(\nu+1/2)\Gamma(1/2)}{\Gamma(\nu+1)}. \tag{14}$$

Equation (14) can also be written in terms of a Beta-function; namely,

$$\frac{\Gamma(\nu+1/2)\Gamma(1/2)}{\Gamma(\nu+1)} = B\left(\nu+\frac{1}{2}, \frac{1}{2}\right), \tag{14a}$$

since $B(p, q) = [\Gamma(p)\Gamma(q)]/[\Gamma(p + q)]$. Thus:

$$I_1 = \sum_{\nu=0}^{\infty} c_\nu \frac{1}{2^\nu} \frac{\Gamma(\nu + 1/2)\Gamma(1/2)}{\Gamma(\nu + 1)} z^\nu = \sum_{\nu=0}^{\infty} z^\nu. \quad . \tag{14b}$$

From Eq. (14b) we find:

$$c_\nu = 2^\nu \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1/2)\Gamma(1/2)}. \tag{15}$$

In the first integral I_1 of Eq. (11) the simple form of the power series ($I_1 = \sum_{\nu=0}^{\infty} z^\nu$) will be used, whereas the form (11b) together with the values of the coefficient, found from Eq. (15), will be used in the second, third, etc. terms in the series (11). For the second term in the series (11), we must consider an expression of the form:

$$I_2 = \int_{-1}^{+1} t^2 z f\left[\frac{z(1 - t^2)}{2}\right](1 - t^2)^{-1/2}\, dt. \tag{16}$$

Substitute in Eq. (16) for $f(z, t)$ the series (11b) together with the values of the coefficients, calculated from Eq. (15):

$$\begin{aligned} I_2 &= \int_{-1}^{+1} t^2 z \sum_{\nu=0}^{\infty} c_\nu \frac{z^\nu (1 - t^2)^\nu}{2^\nu} (1 - t^2)^{-1/2}\, dt \\ &= \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1/2)\Gamma(1/2)} z^{\nu+1} \int_{-1}^{+1} t^2 (1 - t^2)^{\nu - 1/2}\, dt \\ &= \sum_{\nu=0}^{\infty} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1/2)\Gamma(1/2)} z^{\nu+1} \frac{\Gamma(\nu + 1/2)\Gamma(3/2)}{\Gamma(\nu + 2)} \\ &= \sum_{\nu=0}^{\infty} \frac{\Gamma(3)}{2^2\Gamma(2)} \frac{z^{\nu+1}}{(\nu + 1)} = \sum_{\nu=0}^{\infty} \frac{1}{2} \frac{z^{\nu+1}}{(\nu + 1)}. \end{aligned} \tag{16a}$$

The third term can be evaluated in a similar manner. We easily obtain:

$$\begin{aligned} \int_{-1}^{+1} t^4 z^2 f\left[\frac{z(1 - t^2)}{2}\right](1 - t^2)^{-1/2}\, dt &= \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1/2)\Gamma(1/2)} z^{\nu+2} \frac{\Gamma(\nu + 1/2)\Gamma(5/2)}{\Gamma(\nu + 3)} \\ &= \frac{\Gamma(5)}{2^4\Gamma(3)} \frac{z^{\nu+2}}{(\nu + 1)(\nu + 2)} = \frac{3}{4} \frac{z^{\nu+2}}{(\nu + 1)(\nu + 2)}, \quad \text{etc.} \end{aligned} \tag{17}$$

Of course, the functions $P^{(n)}$ will be functions of z^* alone, if expressions (12) to (17) are functions of z alone and vice versa. Taking all four sequences into account, we find:

$$\begin{aligned} U(z, z^*) = &\sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} K_{n\nu} P^{(I1n)}(z^*) z^{n+\nu} \\ &+ \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} K_{n\nu} P^{(I2n)}(z^*) z^{n+\nu} \\ &+ \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} K_{n\nu} P^{(II1n)}(z) z^{*n+\nu} \\ &+ \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} K_{n\nu} P^{(II2n)}(z) z^{*n+\nu}, \end{aligned} \tag{18}$$

where

$$
\begin{aligned}
K_{0\nu} &= 1, \quad & K_{10} &= \frac{1}{2}\cdot\frac{1}{1}, & K_{20} &= \frac{3}{4}\cdot\frac{1}{1\cdot 2}, & &\cdots,\\
& & K_{11} &= \frac{1}{2}\cdot\frac{1}{2}, & K_{21} &= \frac{3}{4}\cdot\frac{1}{2\cdot 3}, & &\cdots,\\
& & K_{12} &= \frac{1}{2}\cdot\frac{1}{3}, & K_{22} &= \frac{3}{4}\cdot\frac{1}{3\cdot 4}, & &\cdots,\\
& & \cdots\cdots &, & \cdots\cdots\cdots &, & &\cdots,\\
& & K_{1\nu} &= \frac{1}{2}\frac{1}{(\nu+1)}, & K_{2\nu} &= \frac{3}{4}\frac{1}{(\nu+1)(\nu+2)} & &\cdots.
\end{aligned}
\tag{19}
$$

Certain constant coefficients have to be introduced in the process of integration; these should be defined by the boundary conditions. Separation of the real and imaginary parts leads to the following form of the general solution of the partial differential equation considered:

$$
\begin{aligned}
U(z, z^*) = {} & a_{\nu 1} Re\left[\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} K_{n\nu} P^{(I1n)}(z^*) z^{n+\nu}\right]\\
& + a_{\nu 2} Im\left[\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} K_{n\nu} P^{(I1n)}(z^*) z^{n+\nu}\right]\\
& + b_{\nu 1} Re\left[\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} K_{n\nu} P^{(I2n)}(z^*) z^{n+\nu}\right]\\
& + b_{\nu 2} Im[\cdots] + c_{\nu 1} Re[\cdots] + c_{\nu 2} Im[\cdots]\\
& + d_{\nu 1} Re\left[\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} K_{n\nu} P^{(II2n)}(z) z^{*n+\nu}\right]\\
& + d_{\nu 2} Im\left[\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} K_{n\nu} P^{(II2n)}(z) z^{*n+\nu}\right].
\end{aligned}
\tag{20}
$$

In case Eq. (2) is taken as the starting point of these considerations, one does not need to take into account the conjugate sequences $P^{(II)}$ since both $P^{(I)}$ and $P^{(II)}$ will give the same results. Thus only the sequence $P^{(I)}$ need be calculated.

2. Functions $P^{(I)}$ and $P^{(II)}$. Of course, there are many functions which satisfy conditions (8) and consequently represent the solution. From Eq. (9), we find that

$$D_1[P^{(0)}] = P^{(0)}_{z^*z^*} + MP^{(0)} = 0,$$

from which it follows that

$$P^{(0)} = \cos(M^{1/2} z^*) \text{ or } P^{(0)} = \sin(M^{1/2} z^*), \tag{21}$$

under the assumption that M = constant. The general solution is

$$P^{(I10)} = A_1 \cos(M^{1/2} z^*) + A_2 \sin(M^{1/2} z^*). \tag{22}$$

From Eq. (10) one may easily find that $A_1 = 1$ and $A_2 = 0$. Thus

$$P^{(I10)} = \cos(M^{1/2} z^*), \tag{23}$$

and similarly,

$$P^{(I20)} = A_1 \cos(M^{1/2} z^*) + A_2 \sin(M^{1/2} z^*) = \frac{1}{M^{1/2}} \sin(M^{1/2} z^*), \tag{24}$$

$$\begin{aligned} D_1[P^{(1)}] &= P^{(1)}_{s^*s^*} + MP^{(1)} = -4D_1[P^{(0)}_{s^*}] - 2D_2[P^{(0)}] \\ &= -4[P^{(0)}_{ss^*s^*} + MP^{(0)}_{s^*}] - 2[LP^{(0)}_{s^*} + AP^{(0)}]. \end{aligned} \tag{25}$$

For $P^{(I10)} = \cos(M^{1/2} z^*) = P^{(0)}$ one obtains

$$P^{(0)}_{s^*} = P^{(0)}_{ss^*s^*} = 0, \qquad P^{(0)}_{s^*} = -M^{1/2} \sin(M^{1/2} z^*), \tag{26}$$

$$P^{(1)}_{s^*s^*} + MP^{(1)} = 2L\, M^{1/2} \sin(M^{1/2} z^*) - 2A \cos(M^{1/2} z^*). \tag{27}$$

The homogeneous equation $P^{(1)}_{s^*s^*} + MP^{(1)} = 0$ has the solutions

$$P^{(1)} = u_1 = \cos(M^{1/2} z^*) \qquad \text{and} \qquad P^{(1)} = u_2 = \frac{1}{M^{1/2}} \sin(M^{1/2} z^*).$$

Any conventional method might be used to solve the non-homogeneous equation (27) by finding the particular integral, for example, the method of variation of parameters. One may easily show, however, all the conventional methods have proved to be highly impractical and inconvenient for higher terms of the sequences $P^{(I1n)}$ and $P^{(I2n)}$. For this reason, the method given by Ince [2] will be used. This proved to be shorter beginning with the term $P^{(I12)}$. As an illustration, let us apply this technique to the solution of Eq. (27). We note that the Wronskian

$$\begin{aligned} \Delta(u_1, u_2) &= \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos(M^{1/2} z^*)(M^{1/2})^{-1} \sin(M^{1/2} z^*) \\ -M^{1/2} \sin(M^{1/2} z^*) \cos(M^{1/2} z^*) \end{vmatrix} = 1. \end{aligned} \tag{28}$$

The functions V_1 and V_2 are given by

$$V_1 = -\int \frac{u_2}{\Delta} r(z^*)\, dz^*, \qquad V_2 = \int \frac{u_1}{\Delta} r(z^*)\, dz^*, \tag{28a}$$

where the expression $r(z^*)$ represents the right side of the non-homogeneous equation,

$$r(z^*) = 2LM^{1/2} \sin(M^{1/2} z^*) - 2A \cos(M^{1/2} z^*). \tag{29}$$

After the necessary substitution and calculations are performed one obtains

$$V_1 = \frac{A}{M} \sin^2(M^{1/2} z^*) - Lz^* + \frac{L}{M^{1/2}} \cos(M^{1/2} z^*) \sin(M^{1/2} z^*) + C_1, \tag{30}$$

$$V_2 = L \sin^2(M^{1/2} z^*) - Az^* - \frac{A}{M^{1/2}} \cos(M^{1/2} z^*) \sin(M^{1/2} z^*) + C_2. \tag{31}$$

Finally the general solution is given by

$$y = V_1 u_1 + V_2 u_2 = P^{(1)} = P^{(I11)}. \tag{32}$$

Condition (10) implies that $C_1 = C_2 = 0$; and, finally,

$$P^{(I11)} = \frac{L}{M^{1/2}} \sin (M^{1/2} z^*) - Lz^* \cos (M^{1/2} z^*) - \frac{A}{M^{1/2}} z^* \sin (M^{1/2} z^*). \tag{33}$$

One can easily show by direct substitution into Eq. (27), that Eq. (33) gives the required result. In a similar way, the function $P^{(I21)}$ and all the others can be found. The table, given below, shows the results. In order to obtain the function $P^{(II)}$ one has to replace z^* by z, M by N, D by A and vice versa.

$$P^{(I10)} : \cos (M^{1/2} z^*),$$

$$P^{(I11)} : \frac{L}{M^{1/2}} \sin (M^{1/2} z^*) - Lz^* \cos (M^{1/2} z^*) - \frac{A}{M^{1/2}} z^* \sin (M^{1/2} z^*),$$

$$P^{(I12)} : \frac{AL}{3 M^{1/2}} z^{*2} \sin (M^{1/2} z^*) + \left[\frac{L^2}{6} - \frac{A_2^2}{6M}\right] z^{*2} \cos (M^{1/2} z^*)$$

$$+ \left[\frac{A^2}{6M M^{1/2}} - \frac{L^2}{6 M^{1/2}} + \frac{2N M^{1/2}}{3} - \frac{2C}{3 M^{1/2}}\right] z^* \sin (M^{1/2} z^*)$$

$$+ \left[\frac{AL}{3M} - \frac{2D}{3}\right] z^* \cos (M^{1/2} z^*) + \frac{2D}{3 M^{1/2}} \sin (M^{1/2} z^*),$$

$$P^{(I20)} : \frac{1}{M^{1/2}} \sin (M^{1/2} z^*),$$

$$P^{(I21)} : \frac{A}{M} z^* \cos (M^{1/2} z^*) - \frac{A}{M M^{1/2}} \sin (M^{1/2} z^*) - \frac{L}{M^{1/2}} z^* \sin (M^{1/2} z^*),$$

$$P^{(I22)} : \left[\frac{L^2}{6 M^{1/2}} - \frac{A^2}{6M M^{1/2}}\right] z^{*2} \sin (M^{1/2} z^*) - \frac{AL}{3M} z^{*2} \cos (M^{1/2} z^*)$$

$$+ \left[\frac{AL}{3M M^{1/2}} - \frac{2D}{3 M^{1/2}}\right] z^* \sin (M^{1/2} z^*)$$

$$+ \left[\frac{-L^2}{6M} - \frac{A^2}{2M^2} - \frac{2N}{3} + \frac{2C}{3M}\right] z^* \cos (M^{1/2} z^*)$$

$$+ \left[\frac{L^2}{6M M^{1/2}} + \frac{A^2}{2M^2 M^{1/2}} + \frac{2N}{3 M^{1/2}} - \frac{2C}{3M M^{1/2}}\right] \sin (M^{1/2} z^*),$$

$$P^{(II10)} : \cos(N^{1/2} z),$$

$$P^{(II11)} : \frac{L}{N^{1/2}} \sin(N^{1/2} z) - Lz \cos(N^{1/2} z) - \frac{D}{N^{1/2}} z \sin(N^{1/2} z),$$

$$P^{(II12)} : \frac{DL}{3N^{1/2}} z^2 \sin(N^{1/2} z) + \left[\frac{L^2}{6} - \frac{D^2}{6N}\right] z^2 \cos(N^{1/2} z)$$
$$+ \left[\frac{D^2}{6N N^{1/2}} - \frac{L^2}{6 N^{1/2}} + \frac{2M N^{1/2}}{3} - \frac{2C}{3 N^{1/2}}\right] z \sin(N^{1/2} z)$$
$$+ \left[\frac{DL}{3N} - \frac{2A}{3}\right] z \cos(N^{1/2} z) + \frac{2A}{3 N^{1/2}} \sin(N^{1/2} z),$$

$$P^{(II20)} : \frac{1}{N^{1/2}} \sin(N^{1/2} z),$$

$$P^{(II21)} : \frac{D}{N} z \cos(N^{1/2} z) - \frac{D}{N N^{1/2}} \sin(N^{1/2} z) - \frac{L}{N^{1/2}} z \sin(N^{1/2} z),$$

$$P^{(II22)} : \left[\frac{L^2}{6 N^{1/2}} - \frac{D^2}{6N N^{1/2}}\right] z^2 \sin(N^{1/2} z) - \frac{DL}{3N} z^2 \cos(N^{1/2} z)$$
$$+ \left[\frac{DL}{3N N^{1/2}} - \frac{2A}{3 N^{1/2}}\right] z \sin(N^{1/2} z)$$
$$+ \left[\frac{-L^2}{6N} - \frac{D^2}{2N^2} - \frac{2M}{3} + \frac{2C}{3N}\right] z \cos(N^{1/2} z)$$
$$+ \left[\frac{L^2}{6N N^{1/2}} + \frac{D^2}{2N^2 N^{1/2}} + \frac{2M}{3 N^{1/2}} - \frac{2C}{3N N^{1/2}}\right] \sin(N^{1/2} z).$$

Direct substitution into Eq. (18) of the expressions for the sequences $P^{(I1n)}$, $P^{(I2n)}$, $P^{(II1n)}$, $P^{(II2n)}$, taken with a limited number of terms, and subsequent substitution of Eq. (18) into (1) show quite easily that all terms with the power z^ν or less than z^ν, i.e. $z^{\nu-1}$, $z^{\nu-2}$ etc. cancel. The terms with the power $z^{\nu+1}$ cancel only partly; i.e. some terms do not drop out. The higher the power of $z^{\nu+p}$, the greater the number of terms which do not drop out. A preliminary test of convergence of the functions $P^{(\nu)}$ should be made at the beginning for values of M, L, C, D, N and A close to those which may be used in the given problem. In the case considered one obtains for $M = L = C = D = N = 1$, $A = 3$ after the necessary calculations are performed:

z^*	0	0.1	0.2	0.3	0.6
$P^{(I10)}$	1	0.995000	0.980100	0.955300	0.825300
$P^{(I11)}$	0	−0.029640	−0.116540	−0.257000	−0.946860
$P^{(I12)}$	0	0.100698	0.239742	0.367548	0.987782

As one can see, the convergence is fairly good for small values of z^*.

3. Partial differential equations of loaded plates with small deflections. The partial differential equation solved by Bergman [1] is homogeneous, i.e., all forms contain the function U. On the other hand the partial differential equations of loaded plates with small deflections are mostly non-homogeneous, i.e., they contain a term without the function U. In these cases one has to perform the transformation of a non-homogeneous equation into a homogeneous one, a task which is not always possible with the use of mathematical methods available today. Let us consider several cases [3]:

(a) laterally loaded isotropic plate:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}, \tag{34}$$

where

q, intensity of a continuously distributed load,

w, deflection,

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

E, Young's modulus,

h, thickness of the plate,

ν, Poisson's ratio.

(b) laterally loaded anisotropic plate:

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = q, \qquad \text{where} \tag{35}$$

$$D_x = \frac{E_x' h^3}{12}, \qquad D_y = \frac{E_y' h^3}{12}, \qquad D_1 = \frac{E'' h^3}{12}, \qquad D_{xy} = \frac{G h^3}{12},$$

$$H = D_1 + 2D_{xy}.$$

(c) isotropic plate bent by moments distributed along the edges:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0; \tag{36}$$

(d) isotropic plate on an elastic foundation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q - kw}{D}, \qquad \text{where} \tag{37}$$

k = modulus of the foundation (pound per square inch per inch of deflection).

(e) isotropic plate bent under the combined action of lateral loads and forces in the middle plane of the plate:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left[q + N_x\frac{\partial^2 w}{\partial x^2} + N_y\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\right], \tag{38}$$

where

N_x, N_y, normal forces per unit length of sections of the plate perpendicular to x- and y-directions, respectively,

N_{xy}, shearing force in the direction of y-axis per unit length of section of the plate perpendicular to x-axis.

Setting $q = 0$ in Eq. (38), we obtain the equation for the deflection of a plate under the action of forces in the middle plane only.

(f) isotropic plate with small initial curvature:

$$\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} = \frac{1}{D}\left[q + N_x \frac{\partial^2(w_0 + w_1)}{\partial x^2} + N_y \frac{\partial^2(w_0 + w_1)}{\partial y^2} + 2N_{xy} \frac{\partial^2(w_0 + w_1)}{\partial x\, \partial y}\right], \tag{39}$$

where w_0, initial deflection,

w_1, additional deflection.

In general we may write

$$\Delta\Delta w + aw_{xx} + 2bw_{xy} + cw_{yy} + fw = X, \tag{40}$$

where X, which is equal in most cases to q/D, may be a constant, or a function of x and y, or sometimes is equal to zero. The general procedure is the following: one finds any particular solution of equation (40). Let it be denoted by w_1. After substituting it into (40) and subtracting this new equation from (40) one obtains the homogeneous equation,

$$\Delta\Delta(w - w_1) + a(w_{xx} - w_{1xx}) + 2b(w_{xy} - w_{1xy}) + c(w_{yy} - w_{1yy}) + f(w - w_1) = 0. \tag{41}$$

Let us consider some special cases:

(a) assume f = const. and X = const. Put w_1 = const. = E; $fE = X$, $w_1 = X/f$,

(b) assume $f = 0$, and X, a, b, c are constant. Put $w_1 = \frac{1}{2}$ const. x^2, w_{1xx} = const., $a \cdot$ const. = X, or $w_1 = \frac{1}{2}(X/a)\, x^2$,

(c) assume $a = b = c = f = 0$, X = const.; put $w_1 = (1/24)$ const. x^4,

$$\frac{\partial^4 w_1}{\partial x^4} = \text{const.} = X, \qquad \text{or} \qquad w_1 = \frac{X}{24} x^4.$$

(d) assume $L(w) = X$ where X is a certain polynomial in x and y with constant coefficients; assume also a, b, c, constant. Thus $X = \sum a_{\nu\mu} x^\nu y^\mu$ or $X = \sum a_{\nu\mu} \cos(\nu x) \cos(\mu y)$. Let w_1 be a polynomial in x and y of the same degree as X. Thus $w_1 = \sum A_{\nu\mu} x^\nu y^\mu$ or $w_1 = \sum A_{\nu\mu} \cos(\nu x) \cos(\mu y)$. Substitution of w_1 and subsequent comparison of the coefficients on both sides yield the values of $A_{\nu\mu}$ in terms of $a_{\nu\mu}$.

In case no such combinations like those given above can be applied, or when the coefficients a, b, c, f, X, are not all constant, one must refer to the general theory of partial differential equations in order to find any particular solution or any fundamental integral. The application of Green's method may be successful in some cases. The particular case $\Delta\Delta w = f$ as well as the method of finding the fundamental integral in the case of an arbitrary linear partial differential equation with constant coefficients are treated in Reference 4. When the equation is homogeneous, the method of linear differential operators explained above can be applied and the deflection of the plate presented in the form (20).

4. Illustration of the method. Calculation of the Critical Stresses. (*a*) *Formulation of the problem and explanation of the procedure.* As an illustration, this method will be applied to the problem of calculating the critical stresses in an isotropic triangular plate subject to a pure uniform compression parallel to the horizontal axis. The triangle is assumed to have a right angle and both legs are equal to l. One of the legs coincides with the x_1-axis and the other with the y_1-axis. In this case, Eq. (38) will reduce to the form

$$\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2\,\partial y_1^2} + \frac{\partial^4 w}{\partial y_1^4} + \tau\frac{\partial^2 w}{\partial x_1^2} = 0, \qquad \text{where} \qquad \tau = \frac{N_x}{D}. \tag{42}$$

Let us transfer the figure into an x, y-system of coordinates such that the length of a leg equals 0.6. Keeping in mind that

$$x = \frac{0.6x_1}{l}, \qquad \frac{\partial w}{\partial x_1} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x_1} = \frac{0.6}{l}\frac{\partial w}{\partial x}, \qquad \frac{\partial^2 w}{\partial x_1^2} = \left(\frac{0.6}{l}\right)^2\frac{\partial^2 w}{\partial x^2}, \qquad \text{etc.},$$

one obtains in the x, y-system the following equation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{l^2}{0.36}\tau\frac{\partial^2 w}{\partial x^2} = 0. \tag{43}$$

In case the two sides are not equal, one has to express the length of one side in terms of the length of the other, and Eq. (43) will remain the same. To each value of N_x = const., there corresponds a set of solutions w satisfying Eq. (43); but the boundary conditions need not be satisfied. However, the boundary conditions can be written in terms of a minimal problem, and the solution w so determined as to satisfy this minimal problem. This method is explained in detail below.

In order to find the first critical stress one has to find the least value from this set of w. To this end, we must vary N_x and look for that value of N_x for which the deflection w reaches its minimum value. Suppose now that τ is multiplied by a number m. In order not to change the character of Eq. (43), we shall introduce a new system of coordinates

$$\zeta = x\,m^{1/2},\ \eta = y\,m^{1/2},\ \frac{\partial w}{\partial x} = \frac{\partial w}{\partial \zeta}\,m^{1/2},\ \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial \zeta^2}\,m,\ \frac{\partial^4 w}{\partial x^4} = \frac{\partial^4 w}{\partial \zeta^4}\,m^2,\ \text{etc.}$$

Eq. (43) will be transformed into

$$\frac{\partial^4 w}{\partial \zeta^4} + 2\frac{\partial^4 w}{\partial \zeta^2\,\partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4} + \frac{l^2}{0.36}\tau\frac{\partial^2 w}{\partial \zeta^2} = 0. \tag{44}$$

Thus multiplication of τ by m is equivalent to the multiplication of the coordinates x and y by $m^{1/2}$ or to the introduction of new coordinates ζ and η. Assuming clamped edges all along the circumference leads to two conditions:

(i) deflection along the circumference equal to zero,

$$\oint_c w\,d\sigma = 0 \qquad \text{or} \qquad \oint_c w^2\,d\sigma = 0, \qquad d\sigma = (d\zeta^2 + d\eta^2)^{1/2}, \tag{45}$$

(ii) the first derivative of the deflection with respect to the normal along the circumference is equal to zero,

$$\oint_c \frac{dw}{dn}\, d\sigma = 0, \qquad \text{or} \qquad \oint_c \left(\frac{dw}{dn}\right)^2 d\sigma = 0. \tag{46}$$

If the solution is not very far from the state where (i) and (ii) are equal to zero, both conditions can be combined into

$$L = \oint_c w^2\, d\sigma + \Lambda \oint_c \left(\frac{dw}{dn}\right)^2 d\sigma = \min. \tag{47}$$

Concerning the value of Λ, some authors (References 5, 6, 7) assume that it is connected with the question of what ratio each of the two conditions $w = 0$ and $dw/dn = 0$ should be taken into account. If the function w satisfies the boundary precisely, both w and dw/dn would vanish on the boundary. The particular solution chosen, however, does not satisfy the boundary conditions completely. That part of the work which arises due to the deviation from $w = 0$ may be expressed as

$$k_1 = C_1 \oint_c w^2\, d\sigma \tag{48}$$

and that part of the work which arises due to the deviation from $dw/dn = 0$ may be expressed as

$$k_2 = C_2 \oint_c \left(\frac{dw}{dn}\right)^2 d\sigma \tag{49}$$

where C_1 and C_2 depend on the properties of boundary supports. The next condition imposed is the requirement that

$$k_1 \cong k_2\,, \qquad \text{i.e.,} \qquad C_1 \oint_c w^2\, d\sigma = C_2 \oint_c \left(\frac{dw}{dn}\right)^2 d\sigma. \tag{50}$$

Therefore, the coefficient Λ is

$$\Lambda = C_2/C_1\,. \tag{51}$$

This requirement implies that the work of the boundary forces which arises from the chosen particular solution divides uniformly into both parts (References 5, 6). For the boundary conditions $w = 0$, $d^2w/dn^2 = 0$, Bergman cites that on the basis of approximate calculations the requirement (50) is fulfilled best if one chooses $\Lambda = 0.002$, 0.5, or 200. In the case considered in this paper, the value $\Lambda = 1$ was chosen after some approximate calculations.

Equation (47) can also be written in terms of the (x, y)-coordinates in the following manner,

$$d\sigma = (d\zeta^2 + d\eta^2)^{1/2} = (m\, dx^2 + m\, dy^2)^{1/2} = m^{1/2}\, ds, \tag{52}$$

$$L = m^{1/2} \oint_c w^2\, ds + \Lambda\, m^{1/2} \oint_c \left(\frac{dw}{dn}\right)^2 ds = \min.$$

One more condition must be superimposed on the whole system, namely, that the energy of the deflected plate should be positive. In the case considered, it is sufficient to assume that the energy

$$E_1 = \frac{1}{2}\iint_s N_x\left(\frac{dw}{d\zeta}\right)^2 d\zeta\, d\eta > 0 \qquad \text{or} \qquad = \text{const.}, \tag{53}$$

Since $N_x = \text{const.}$ this condition may be expressed as

$$E = \iint_s \left(\frac{dw}{d\zeta}\right)^2 d\zeta\, d\eta = m\iint_s \left(\frac{dw}{m^{1/2}\,dx}\right)^2 dx\, dy > 0 \qquad \text{or} \qquad = \text{const.} \tag{54}$$

It is obvious that in Eqs. (44) to (50), (53) and (54), w is expressed in ζ, η system of coordinates. In the case considered $m = 1, 4, 9, 16, 25$ and 36 was chosen.

The deflection w will be expressed by $U(z, z^*)$ in Eq. (20). Since Eq. (42) was chosen as the starting point, only the sequences $P^{(I1)}$ and $P^{(I2)}$ will be taken into account. After all necessary calculations are completed, the imaginary part of $U(z, z^*)$ will be assumed to be real, since one may easily see from Bergman's method [1] that for real partial differential equations the function u in $U = u + iv$ will be real. The circumference of the triangle and its surface will be divided into several parts. In the case of a curvilinearly shaped plate, the circumference has to be transformed to a polygon and the precision of the calculation will depend on the number of sides. The coordinates x and y are measured for the center of each part or side and the sequences $P^{(I1)}$ and $P^{(I2)}$ are calculated for a finite number of terms by substitution of these measured values of x and y. The unknown coefficients $a_{\nu 1}$, $a_{\nu 2}$, $b_{\nu 1}$, $b_{\nu 2}$, are determined from the minimal problem with constraints discussed previously. This leads to a set of n linear equations with n unknowns of the form

$$\frac{\partial L}{\partial a_{\nu 1}} + \lambda\frac{\partial E}{\partial a_{\nu 1}} = 0, \tag{55}$$

where λ is the Lagrangian multiplier. These equations will possess solutions $a_{\nu 1}$ and $b_{\nu 1}$ different from zero if and only if the determinant of the coefficients vanishes; that is, provided

$$\begin{vmatrix} e_{11} + \lambda g_{11} , & \cdots\cdots\cdots & e_{1n} + \lambda g_{1n} \\ \vdots & & \vdots \\ e_{n1} + \lambda g_{n1} , & \cdots\cdots\cdots & e_{nn} + \lambda g_{nn} \end{vmatrix} = F(\lambda) = 0. \tag{56}$$

In order to find the first value of w one does not need to calculate the deflection from Eq. (20). It is quite enough to take $F(\lambda)$ into account (Reference 8)*, for consider the pair of quadratic forms

$$\begin{aligned} \Phi(x_1 , \cdots\cdots x_n) &= \sum_1^n a_{ij}x_i x_j , \\ \psi(x_1 , \cdots\cdots x_n) &= \sum_1^n b_{ij}x_i x_j . \end{aligned} \tag{57}$$

*Chapter 13, p. 169, Theorem 3.

Then the function

$$\Phi - \lambda\psi = \sum_1^n (a_{ij} - \lambda b_{ij}) x_i x_j \,, \tag{58}$$

has as its discriminant, the determinant

$$\begin{vmatrix} a_{11} - \lambda b_{11} \,, & \cdots\cdots\cdots & a_{1n} - \lambda b_{1n} \\ \vdots & & \vdots \quad \vdots \\ a_{n1} - \lambda b_{n1} \,, & \cdots\cdots\cdots & a_{nn} - \lambda b_{nn} \end{vmatrix} = F(\lambda). \tag{59}$$

If $\lambda_1 \,, \lambda_2 \,, \lambda_3 \cdots \lambda_n$ denote the roots of the λ-equation of the pair of forms Φ and ψ, $F(\lambda) = 0$, then it can be proved that Φ and ψ may be reduced by means of a non-singular linear transformation to the normal forms

$$\begin{gathered} \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots\cdots + \lambda_n x_n^2 \,, \\ x_1^2 + \quad x_2^2 + \cdots\cdots + \quad x_n^2 \,. \end{gathered} \tag{60}$$

Thus we obtain a pair of quadratic forms, one with coefficients $\lambda_1 \,, \lambda_2 \,, \lambda_3 \cdots \lambda_n$, and the other with all coefficients equal to 1. Hence we may consider the pair of quadratic forms (60) instead of (57). Since one looks for the first value, only λ_1 has to be taken into account for various values of m. The minimum of the function $\lambda_1 = f(m)$, which may be plotted as a diagram, gives the value of m for the critical stress.

(b) *Calculations of the sequences* $K_{n\nu}P^{(I1n)}(z^*)z^{n+\nu}$ *and* $K_{n\nu}P^{(I2n)}(z^*)z^{n+\nu}$. Let us take the first three terms of the first sequence (18):

$$\begin{aligned} U(z, z^*)_1 = P^{(I10)}(z^*)z^\nu + \frac{1}{2} P^{(I11)}(z^*) \frac{z^{\nu+1}}{(\nu+1)} \\ + \frac{3}{4} P^{(I12)}(z^*) \frac{z^{\nu+2}}{(\nu+1)(\nu+2)}. \end{aligned} \tag{61}$$

If we substitute (23), (33) etc. into (61), comparison of (2) and (43) gives:

$$\begin{gathered} a = \frac{l^2}{0.36} \frac{N_x}{D}, \qquad b = c = d = e = f = 0, \qquad M_2 = A_1 = A_2 = C = 0, \\ L = \frac{a}{8} = \frac{l^2}{0.36} \frac{N_x}{8D}, \qquad M_1 = \frac{a}{16} = \frac{l^2}{0.36} \frac{N_x}{16D}, \\ N = \frac{a}{16} = \frac{l^2}{0.36} \frac{N_x}{16D} = \frac{l^2 N_x}{5.76D}, \end{gathered} \tag{62}$$

$A = D = C = 0.$

Let us assume in (1) that $M = 1$, $L = 2$, $N = 1$ and substitute these values into (23), (33) etc. and into (61). After necessary calculations have been made we obtain

$$\begin{aligned} U(z, z^*)_1 = (\cos z^*)z^\nu + (\sin z^* - z^* \cos z^*) \frac{z^{\nu+1}}{(\nu+1)} \\ + \frac{1}{2} z^{*2} (\cos z^*) \frac{z^{\nu+2}}{(\nu+2)(\nu+1)}. \end{aligned} \tag{63}$$

Let us assume $\nu = 0, 1, 2, 3$, and decompose (63) into the real and the imaginary parts, calling the real part $\psi_{\nu 1}$, and the imaginary $\psi_{\nu 2}$. Then

$$\psi_{01} = x \sin x \cosh y + y \cos x \sinh y + \left[\frac{x^4 + y^4}{4} + \frac{x^2y^2}{2} - x^2 - y^2 + 1\right]\cos x \cosh y, \tag{64}$$

$$\psi_{02} = -ix \cos x \sinh y + iy \sin x \cosh y + \left[\frac{x^4 + y^4}{4} + \frac{x^2y^2}{2} - x^2 - y^2 + 1\right] i \sin x \sinh y. \tag{65}$$

It can be easily seen that $\psi_{\nu 2}$ can be obtained from $\psi_{\nu 1}$ changing $(\sin x)$ into $(-i \cos x)$, $(\cos x)$ into $(i \sin x)$, $(\sinh y)$ into $\cosh y)$ and $(\cosh y)$ into $(\sinh y)$. Consequently, it is enough to give the results for $\psi_{\nu 1}$ and $\Phi_{\nu 1}$ only.

$$\psi_{11} = \left[\frac{x^5}{12} + \frac{x^3y^2}{6} + \frac{y^4x}{12} - \frac{x^3}{2} - \frac{xy^2}{2} + x\right]\cos x \cosh y - \left[\frac{x^4y}{12} + \frac{x^2y^3}{6} + \frac{y^5}{12} - \frac{y^3}{2} - \frac{x^2y}{2} + y\right]\sin x \sinh y + \left[\frac{x^2}{2} - \frac{y^2}{2}\right]\sin x \cosh y + xy \cos x \sinh y, \tag{66}$$

$$\psi_{21} = \left[\frac{x^6 + x^4y^2 - x^2y^4 - y^6}{24} + \frac{x^4 - y^4}{3} + x^2 - y^2\right]\cos x \cosh y - \left[\frac{x^5y + xy^5}{12} + \frac{x^3y^3}{6} + \frac{2x^3y + 2xy^3}{3} + 2xy\right]\sin x \sinh y + \frac{x^3 - 3xy^2}{3}\sin x \cosh y + \frac{3x^2y - y^3}{3}\cos x \sinh y, \tag{67}$$

$$\psi_{31} = \left[\frac{x^7 - x^5y^2 - 5x^3y^4 - 3xy^6}{40} - \frac{x^5 - 2x^3y^2 - 3xy^4}{4} + x^3 - 3xy^2\right]\cos x \cosh y - \left[\frac{3x^6y + 5x^4y^3 + x^2y^5 - y^7}{40} - \frac{3x^4y + 2x^2y^3 - y^5}{4} + 3x^2y - y^3\right]\sin x \sinh y + \frac{x^4 - 6x^2y^2 + y^4}{4}\sin x \cosh y + (x^3y - xy^3)\cos x \sinh y. \tag{68}$$

In the same way the function $U(z, z^*)_2$ for $P^{(I2)}$ can be calculated; namely,

$$U(z, z^*)_2 = P^{(I20)}(z^*)z^\nu + \frac{1}{2}P^{(I21)}(z^*)\frac{z^{\nu+1}}{(\nu+1)} + \frac{3}{4}P^{(I12)}(z^*)\frac{z^{\nu+2}}{(\nu+1)(\nu+2)}$$

$$= (\sin z^*)z^\nu - z^*(\sin z^*)\frac{z^{\nu+1}}{(\nu+1)} \tag{69}$$

$$+ \left(\sin z^* - z^*\cos z^* + \frac{1}{2}z^{*2}\sin z^*\right)\frac{z^{\nu+2}}{(\nu+1)(\nu+2)}.$$

Let us call the real part $\Phi_{\nu 1}$, and the imaginary $\Phi_{\nu 2}$. Then

$$\Phi_{01} = \left[\frac{x^4 + 2x^2y^2 + y^4}{4} - \frac{x^2 + 3y^2}{2} + 1\right]\sin x\cosh y$$

$$+ xy\cos x\sinh y - \frac{x^3 + xy^2}{2}\cos x\cosh y + \frac{x^2y + y^3}{2}\sin x\sinh y, \tag{70}$$

$$\Phi_{11} = \left[\frac{x^5 + 2x^3y^2 + xy^4}{12} - \frac{x^3 + 3xy^2}{3} + x\right]\sin x\cosh y$$

$$+ \left[\frac{x^4y + 2x^2y^3 + y^5}{12} - \frac{2y^3}{3} + y\right]\cos x\sinh y \tag{71}$$

$$- \frac{x^4 - y^4}{6}\cos x\cosh y + \frac{x^3y + xy^3}{3}\sin x\sinh y,$$

$$\Phi_{21} = \left[\frac{x^6 + x^4y^2 - x^2y^4 - y^6}{24} - \frac{5y^4 - 3x^4}{12} + x^2 - y^2\right]\sin x\cosh y$$

$$+ \left[\frac{x^5y + 2x^3y^3 + xy^5}{12} - \frac{x^3y + 3xy^3}{3} + 2xy\right]\cos x\sinh y \tag{72}$$

$$- \frac{x^5 - 2x^3y^2 - 3xy^4}{12}\cos x\cosh y + \frac{3x^4y + 2x^2y^3 - y^5}{12}\sin x\sinh y,$$

$$\Phi_{31} = \left[\frac{x^7 - x^5y^2 - 5x^3y^4 - 3xy^6}{40} + \frac{15xy^4 - x^5 - 10x^3y^2}{30} + x^3 - 3xy^2\right]\sin x\cosh y$$

$$+ \left[\frac{3x^6y + 5x^4y^3 + x^2y^5 - y^7}{40}\right.$$

$$\left. + \frac{2y^5 - 10x^2y^3}{30} + 3x^2y - y^3\right]\cos x\sinh y \tag{73}$$

$$- \left[\frac{x^6 - 5x^4y^2 - 5x^2y^4 + y^6}{20}\right]\cos x\cosh y + \frac{x^5y - xy^5}{5}\sin x\sinh y.$$

(c) *Graphical representation of the particular solutions.* A graphical representation for the particular solutions can be obtained in the following manner:

$$\text{Let } z = re^{i\phi}, \qquad z^\nu = r^\nu (\cos \nu\phi + i \sin \nu\phi),$$

$$z^* = re^{-i\phi}, \qquad z^{*\nu} = r^\nu (\cos \nu\phi - i \sin \nu\phi).$$

The values of $\psi_{\nu n}$ and $\Phi_{\nu n}$ can be calculated for some number of points, say for $r = 0.2$, 0.4, 0.6, 0.8, 1.0 and $\phi = 15°, 30°, 45° \cdots 90°$. These points are plotted on a polar diagram r, ϕ separately for $\psi_{\nu 1}$ and $\Phi_{\nu 2}$. Points of the equal value of $\psi_{\nu n}$ and $\Phi_{\nu n}$ can be obtained by interpolation and the lines connecting these points will give the graphical representation of the particular solutions $\psi_{\nu n}$ and $\Phi_{\nu n}$. Some of the results obtained in this way are shown in Fig. 1. Similar graphs can be obtained for $P^{(IIn)}$. The real part of $P^{(In)}$ will be the same as that of $P^{(IIn)}$ and the imaginary part of $P^{(IIn)}$ will have the same magnitude but the opposite sign to that of $P^{(In)}$.

(d) *Calculation of derivatives.* The derivative dw/dn has to be calculated for Eq. (52):

$$\frac{dw}{dn} = \frac{d\psi_{\nu n}}{dn} = \frac{\partial \psi_{\nu n}}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial \psi_{\nu n}}{\partial y}\frac{\partial y}{\partial n} = \frac{1}{2^{1/2}}\left[\frac{\partial \psi_{\nu n}}{\partial x} + \frac{\partial \psi_{\nu n}}{\partial y}\right], \tag{74}$$

since $\partial x/\partial n = \cos \alpha = 1/2^{1/2}$, $\partial y/\partial n = \sin \alpha = 1/2^{1/2}$, $\alpha = 45°$.

For the horizontal and the vertical sides Eq. (74) changes to $d\psi_{\nu n}/dn = (d\psi_{\nu n}/dy)_{y=0}$ and $d\psi_{\nu n}/dn = (d\psi_{\nu n}/dx)_{x=0}$ respectively. For Eq. (54) $d\psi_{\nu n}/dx$, $(d\psi_{\nu n}/dx)_{x=0}$ and $(d\psi_{\nu n}/dx)_{y=0}$ must be calculated. The results are given below:

$$\frac{\partial \psi_{01}}{\partial x} = [x^3 + xy^2 - x]\cos x \cosh y - \left[\frac{x^4 + y^4}{4} + \frac{x^2y^2}{2} - x^2 - y^2\right]\sin x \cosh y - y \sin x \sinh y, \tag{75}$$

$$\frac{\partial \psi_{01}}{\partial y} = [x^2y + y^3 - y]\cos x \cosh y + \left[\frac{x^4 + y^4}{4} + \frac{x^2y^2}{2} - x^2 - y^2 + 2\right]\cos x \sinh y + x \sin x \sinh y, \tag{76}$$

$$\frac{d\psi_{01}}{dn} = \frac{1}{2^{1/2}}\left[(x^3 + xy^2 - x + x^2y + y^3 - y)\cos x \cosh y - \left(\frac{x^4 + y^4}{4} + \frac{x^2y^2}{2} - x^2 - y^2\right)\sin x \cosh y + \left(\frac{x^4 + y^4}{4} + \frac{x^2y^2}{2} - x^2 - y^2 + 2\right)\cos x \sinh y + (x - y)\sin x \sinh y\right], \tag{77}$$

To obtain the derivatives of ψ_{02} the same changes as explained above have to be introduced. In a similar way, all the necessary derivatives $\partial\psi_{11}/\partial x$, $\partial\psi_{11}/\partial y$, etc., $\partial\phi_{01}/\partial x$, $\partial\phi_{01}/\partial y$, etc. were calculated.

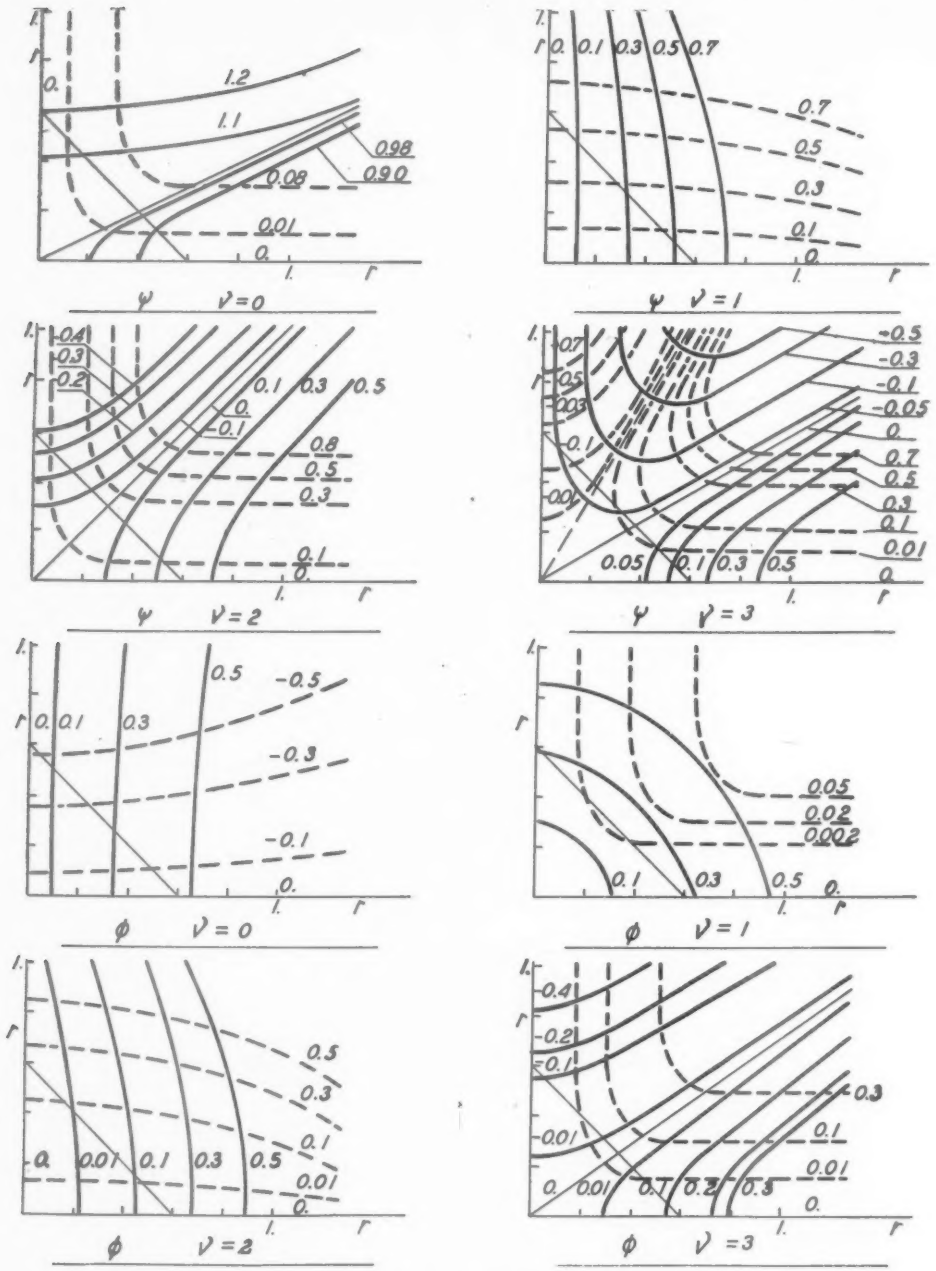

FIG. 1. The functions of ψ and ϕ in polar coordinates (r, φ).
———Real part — — — — Imaginary part

(e) *Procedure for calculation*—It is easily seen that the number of operations involved in the evaluation the above equations may reach a hundred thousand or more. For this reason punch-card-machines were used. The circumference and the surface of the triangle were divided into some number of parts; for each part, the coordinates of the center point were defined and substituted into the above equations for $\psi_{\nu n}$, $\Phi_{\nu n}$ and for all necessary derivatives.

The calculations were performed simultaneously for six values of m, which is equivalent to the multiplication of all terms in x and y of the first degree by $m^{1/2}$ of the second degree by m, of the fourth degree by m^2, etc. Of course, the values of the trigonometric and hyperbolic functions also have to be changed with change in m. In the next step, the squares in (52) and (53) were calculated. This is equivalent to the squaring of 16 term polynomials, since 4 groups $\psi_{\nu 1}$, $\psi_{\nu 2}$, $\Phi_{\nu 1}$, $\Phi_{\nu 2}$ with 4 terms in each group were taken into account. The integrations in Eq. (52) and (53) were replaced by multiplication of each $\psi_{\nu n}$ by the corresponding partial length or surface and summation on the punch-card-machines. Next, the smallest roots of (56) were found in several cases. In order to obtain a good picture of the influence of the order of the determinant on the value of λ, the value of the smallest root, λ, was calculated for determinants of various orders. Determinants were calculated for all six values of m. The results of these calculations are given in Table I for three values of m in the neighborhood of the minimum. All determinants were calculated by the use of punch-card-machines.

TABLE 1. $\lambda = f(m)$

No.	Coefficients Used	Order of determ.	Value of $m = 3^2$	Value of $m = 4^2$	Value of $m = 5^2$
1	a_{01}	1	−54.5	−5.601058	−8.660602
2	a_{01}, b_{01}	2	−2.8	−0.15674675	−0.962
3	a_{01}, a_{02}	2	−8.77342465	−0.163899924	−1.203058734
4	a_{01}, a_{02}, b_{01}, b_{02}	4	−0.0394278	−0.000211601	−0.603463788

To obtain more precise value of λ_{cr} , the values of the two functions ψ_{01} , ψ_{02} were calculated for all integers between 17 and 24. Table I shows that it is sufficient to take into account only the first function ψ_{01} for the location of λ_{cr} . The results of calculations for ψ_{01} are given below.

TABLE 2. $\lambda = f(m)$

m	17	18	19	20
λ	−5.344245	−5.288552	−5.225255	−5.178953
m	21	22	23	24
λ	−5.306672	−5.329772	−5.639540	−6.576221

Thus λ_{cr} is at $m = 20$. To obtain still more precise value of λ_{cr} one may repeat the calculations for several decimal values between 19 and 21.

The value of the critical stress is given by Eq. (62):

$$mN = \frac{ml^2N_x}{5.76D}, \qquad (mN_x) = N_{xcr}, \qquad N = 1,$$

$$N_{xcr} = 20 \times 5.76 \frac{D}{l^2} = 115.2 \frac{D}{l^2}. \tag{78}$$

This may be considered as the first approximation* of N_{xcr}.

This method may also be applied to the case where the coefficients in Eq. (40) are analytic functions in x and y. Since the procedure in this case remains the same as in the case of a partial differential equation of elliptic type and second order one may refer to Reference 10.

The whole procedure may be performed by a graphical method. Diagrams for the particular solutions, for the first and second derivatives, can be plotted as explained above, and the corresponding values read off for each plate shape considered. There is a possibility of preparing a set of such diagrams for various values of forces N_x, N_y, N_{xy}, taken in certain intervals, similar to charts used in practice for buckling of columns. This technique has been used in order to check the results obtained on the punch-card-machines; the agreement was bound to be astonishingly good.

Similarly, in the case where the coefficients $M, L, N \cdots$ in (1) are analytic functions of z and z^*, one must refer to the general method of solution of such equations by use of punch-card-machines [10].

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*For comparison purposes several values are given below: square plate, built-in edges, compression in x-direction, $N_{xcr} = 105\ (D/a^2)$; square plate, simply supported edges, compression in x-direction, $N_{xcr} = 4\pi^2(D/a^2)$; equilateral triangular plate, simply supported edges, uniformly compressed, $N_{cr} = 4\pi^2(D/a^2)$, a = height.

ON A PROBLEM IN PLANE STRAIN

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1. Introduction. The present paper concerns the application to a particular problem in plane elasticity of the method recently developed by W. Prager and J. L. Synge[1] for obtaining approximate solutions to boundary-value problems in elasticity.

The problem considered is that of an infinite bar of rectangular cross-section in a state of plane strain caused by compression as shown in Fig. 1. We assume the conditions of the test to be such that the bar is compressed by a prescribed amount $2a$, the points on the top and bottom faces being permitted vertical (u_2) *but no horizontal* (u_1) *displacement.* Thus, after deformation the cross-section of the bar will appear as in Fig. 2.

This is not a simple compression test. In fact, the exact determination of the stresses

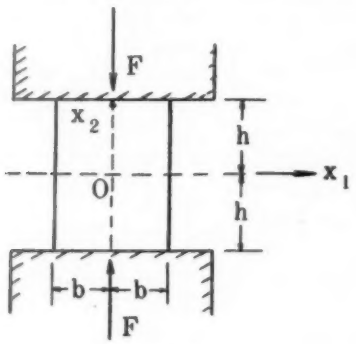

FIG. 1.

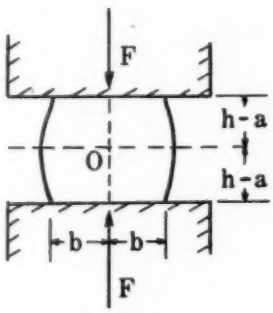

FIG. 2.

in the bar requires the solution of a mixed boundary value problem with the displacements prescribed along $x_2 = \pm h$ and the stresses prescribed (to vanish) along $x_1 = \pm b$.

We propose in the present paper to determine the relation between the force F per unit length of the bar, the compression $2a$ and the Young's modulus E for the material assuming the value of Poisson's ratio ν to be known. (We note that the range of ν for most materials is small as compared with the possible range of E. In any case, by assuming extreme values of ν one may determine the effect of varying ν on the relation between E, F and a.) Knowing this relation, the value of E for a given material can be determined from test measurements of F and a. This test has the advantage that the usual precautions taken to eliminate friction between specimen and end-blocks become unnecessary.

To solve the problem stated above one needs only to determine the strain energy of deformation of the bar. This we shall do to a good approximation by finding upper and lower bounds for this quantity. These bounds can be obtained by means of the Prager-Synge method. To apply this method we must construct *artificial* states of stress

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[1]W. Prager and J. L. Synge, Quarterly Appl. Math. 5, 241-271 (1947).

which approximate to the *natural* state of stress. These artificial states of stress will be discussed in the following section.

2. Specification of states of stress. According to the Prager-Synge method a state of stress, given by a set of six stress components E_{ij} which are functions of position, is looked upon as defining a point or vector **S** in function space. Given two states **S** and **S'**, the *scalar product* of these states is defined to be

$$\mathbf{S}\cdot\mathbf{S}' = \int E_{ij}e'_{ij}\,dv$$

where e'_{ij} are the strains corresponding to the stresses E'_{ij} of the state **S'** and are computed from the E'_{ij} by means of Hooke's law; the integral is extended over the volume of the body. It is easily verified that

$$\mathbf{S}\cdot\mathbf{S}' = \mathbf{S}'\cdot\mathbf{S}$$

so that the scalar product is commutative.

The sum $\mathbf{S} + \mathbf{S}'$ of two states is defined as the state with the stresses $E_{ij} + E'_{ij}$, the product $c\mathbf{S}$ of a state by a scalar c is defined to be the state with the stresses cE_{ij}. It follows that the usual laws of vector algebra remain unchanged.

We shall reserve the notation **S** for the *natural* state of stress E_{ij} in the bar. Thus, the stresses E_{ij} are assumed to satisfy the equilibrium and compatibility equations and in addition all of the boundary conditions of the problem. We note that

$$S^2 = \mathbf{S}\cdot\mathbf{S} = \int E_{ij}e_{ij}\,dv, \tag{2.1}$$

so that *the square of the magnitude of* **S** *is equal to twice the strain energy of deformation of the bar.*

According to the method, we must now select various artificial states of stress which satisfy some but not all of the conditions of the problem. We note first that our boundary conditions specify the displacements on the top and bottom of the bar and zero stresses along the sides. This makes our problem one of a type considered by Prager and Synge and designated by them as Displacement Boundary Condition (DBC). In accordance with their treatment of problems of the DBC type we shall need the following states:

(a) A state $\mathbf{S}^*$ called the *completely associated state.* The stresses E^*_{ij} must satisfy the *compatibility* equation. The displacements defined by the E^*_{ij} are required to satisfy the prescribed boundary conditions on displacements.

(b) A sequence of states $\mathbf{S}'_1$, $\mathbf{S}'_2$, $\cdots$, $\mathbf{S}'_p$ called the *homogeneous associated states.* The stresses ${}_pE'_{ij}$ of the state $\mathbf{S}'_p$ must satisfy the compatibility equations. The displacements defined by the ${}_pE'_{ij}$ are required to vanish wherever displacements are prescribed by the boundary conditions.

(c) A sequence of states $\mathbf{S}''_1$, $\mathbf{S}''_2$, $\cdots$, $\mathbf{S}''_q$ called the *complementary states.* The stresses ${}_qE''_{ij}$ of the state $\mathbf{S}''_q$ must satisfy the *equilibrium* equations and in addition the boundary conditions on stresses.

Since the bar under compression is assumed to be in plane strain, the displacements u_1 and u_2 are functions of x_1 and x_2 alone and there is no displacement perpendicular to the x_1x_2-plane. The boundary conditions to be satisfied by the state $\mathbf{S}^*$ are therefore simply

$$\left.\begin{aligned} u_1^*(x_1\ ,\ \pm h) &= 0 \\ u_2^*(x_1\ ,\ +h) &= -a, \qquad u_2^*(x_1\ ,\ -h) = a, \end{aligned}\right\} \tag{2.2}$$

where u_i^* and u_2^* are the displacements which give rise to the stresses E_{ij}^* of the state $\mathbf{S}^*$. Accordingly, the boundary conditions to be satisfied by the displacements ${}_pu_1'$ and ${}_pu_2'$ associated with the stresses ${}_pE_{ij}'$ of the state $\mathbf{S}_p'$ are

$$\left.\begin{aligned} {}_pu_1'(x_1\ ,\ \pm h) &= 0 \\ {}_pu_2'(x_1\ ,\ \pm h) &= 0. \end{aligned}\right\} \tag{2.3}$$

It is simplest to obtain the associated states by beginning with displacement functions satisfying the boundary conditions, computing from these the strains and finally from the strains computing the stresses by means of Hooke's law. The stresses so obtained are of course automatically compatible.

In plane strain the stress components E_{13} and E_{23} are identically zero and

$$E_{33} = \nu(E_{11} + E_{22}) \tag{2.4}$$

where ν is Poisson's ratio. Since all quantities are independent of the variable x_3, the equilibrium equations take the form

$$\left.\begin{aligned} \frac{\partial E_{11}}{\partial x_1} + \frac{\partial E_{12}}{\partial x_2} &= 0 \\ \frac{\partial E_{21}}{\partial x_1} + \frac{\partial E_{22}}{\partial x_2} &= 0. \end{aligned}\right\} \tag{2.5}$$

Since there are no loads applied to the sides of the bar we have the following boundary conditions on the stresses

$$E_{11}(\pm b,\ x_2) = E_{12}(\pm b,\ x_2) = 0. \tag{2.6}$$

The stresses ${}_qE_{ij}''$ of the state $\mathbf{S}_q''$ are chosen to satisfy (2.5) and (2.6). Only the three functions ${}_qE_{11}''$, ${}_qE_{12}''$ and ${}_qE_{22}''$ need be chosen for each q, since ${}_qE_{21}'' = {}_qE_{12}''$ and we may take ${}_qE_{13}'' = {}_qE_{23}'' = 0$ and ${}_qE_{33}'' = \nu({}_qE_{11}'' + {}_qE_{22}'')$ by virtue of the preceding remarks.

3. Upper and lower bounds for Young's modulus. Prager and Synge gave the following formula for upper and lower bounds on the quantity S^2 which is twice the strain energy of deformation of the natural state:

$$\sum_{q=1}^{n} (\mathbf{S}^* \cdot \mathbf{I}_q'')^2 \leq S^2 \leq S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^* \cdot \mathbf{I}_p')^2; \tag{3.1}$$

the states $\mathbf{I}_p'$ are obtained from the states $\mathbf{S}_p'$ by orthonormalizing these latter states according to the requirement that

$$\mathbf{I}_p' \cdot \mathbf{I}_r' = \begin{cases} 1, & \text{if} \quad p = r \\ 0, & \text{if} \quad p \neq r \end{cases}$$

and the states $\mathbf{I}_q''$ are similarly obtained from the states $\mathbf{S}_q''$. Since the strain energy of deformation is equal to the work done by the external forces we have

$$S^2 = 2Fa \tag{3.2}$$

or in terms of the average normal stress $\sigma = F/2b$ exerted on the bar by the end blocks,

$$S^2 = 4ab\sigma. \tag{3.3}$$

Furthermore, from the definition of the scalar products involved, the conditions on the various states $\mathbf{S}^*$, $\mathbf{S}_p'$, $\mathbf{S}_q''$ and the manner of orthonormalization of the states $\mathbf{S}_p'$ and $\mathbf{S}_q''$ it follows that the upper and lower bounds in (3.1) are proportional to Young's modulus E and to a^2. We can therefore write for (3.1), taking into account (3.2),

$$L_n E a^2 \leq 2Fa \leq U_m E a^2 \tag{3.4}$$

where

$$L_n = \frac{1}{Ea^2} \sum_{q=1}^{n} (\mathbf{S}^* \cdot \mathbf{I}_q'')^2,$$

$$U_m = \frac{1}{Ea^2} \left[S^{*2} - \sum_{p=1}^{m} (\mathbf{S}^* \cdot \mathbf{I}_p')^2 \right]. \tag{3.5}$$

From (3.4) we at once obtain

$$\frac{2F}{aU_m} \leq E \leq \frac{2F}{aL_n} \tag{3.6}$$

or in terms of the average stress σ and the "average strain" $\epsilon = a/h$

$$\frac{4b}{hU_m} \frac{\sigma}{\epsilon} \leq E \leq \frac{4b}{hL_n} \frac{\sigma}{\epsilon}. \tag{3.7}$$

Thus, we obtain upper and lower bounds for Young's modulus in terms of F and a. Alternatively if we are given E and F we can find upper and lower bounds for a or given E and a we can find upper and lower bounds for F. Since L_n increases and U_m decreases as more terms are added in (3.5) (i.e., as n and m are increased) the bounds can be made to differ by a negligible amount so that in effect we determine the numerical coefficient λ in the formula

$$E = \lambda \frac{F}{a}.$$

We note that the quantities L_n and U_m depend on Poisson's ratio ν which we must therefore assume to be known. The exact form of the dependence however can be easily obtained for $n = m = 1$.

4. Selecting the functions and computing the bounds.[2] For convenience we shall take $b = h = 1$ and $\nu = \frac{1}{3}$ in the following. The completely associated state $\mathbf{S}^*$ and the five homogeneous associated states $\mathbf{S}_1'$, $\mathbf{S}_2'$, $\mathbf{S}_3'$, $\mathbf{S}_4'$, $\mathbf{S}_5'$ which we use are given in Table 1. Tabulated are the displacement components, strains and stresses corresponding to each of these states. For convenience we have also set $E = 1$ and $a = 1$; these quantities are easily reintroduced into the final answers by proper proportionality factors.

The complementary states $\mathbf{S}_p''$, $p = 1, 2, \cdots$, are most conveniently defined by means of the Airy stress function ψ. The stresses are then computed from the function ψ by taking

[2] In this work we were ably assisted by Mr. F. Edelman who performed the bulk of the computations.

$$\left.\begin{aligned} E'_{11} &= \frac{\partial^2 \psi}{\partial x_2^2} \\ E'_{22} &= \frac{\partial^2 \psi}{\partial x_1^2} \\ E'_{12} &= -\frac{\partial^2 \psi}{\partial x_1 \, \partial x_2} \end{aligned}\right\} \tag{4.1}$$

When defined in this way, the stresses automatically satisfy the equilibrium equations (2.5) for any choice of the function ψ. We note that not any function ψ may be used, however, since the stresses must satisfy the boundary conditions (2.6) where $b = 1$. The five complementary states we use are given in Table 2.

In orthonormalizing the sequence of states $\mathbf{S}'_1, \cdots, \mathbf{S}'_5$ to obtain the orthonormal sequence $\mathbf{I}'_1, \cdots, \mathbf{I}'_5$ the scalar products $\mathbf{S}'_i \cdot \mathbf{S}'_j$ are required. These are listed in Table 3. To illustrate, we have $\mathbf{S}'_2 \cdot \mathbf{S}'_2 \doteq 4.8$, $\mathbf{S}'_3 \cdot \mathbf{S}'_5 = \mathbf{S}'_5 \cdot \mathbf{S}'_3 = 0.925714$, etc. Also given in Table 3 are the values of the scalar products $\mathbf{S}^* \cdot \mathbf{S}'_j$, $j = 1, \cdots, 5$ and $\mathbf{S}^* \cdot \mathbf{S}^*$ which are needed in the evaluation of the quantities in (3.5).

TABLE 1: The associated states.

State	u'_1	u'_2	$e'_{11} = \dfrac{\partial u'_1}{\partial x_1}$	$e'_{12} = \dfrac{1}{2}\left(\dfrac{\partial u'_1}{\partial x_2} + \dfrac{\partial u'_2}{\partial x_1}\right)$	$e'_{22} = \dfrac{\partial u'_2}{\partial x_2}$
$\mathbf{S}'_1$	$x_1(1 - x_2^2)$	0	$1 - x_2^2$	$-x_1x_2$	0
$\mathbf{S}'_2$	0	$-x_2(1 - x_2^2)$	0	0	$3x_2^2 - 1$
$\mathbf{S}'_3$	$x_1^3(1 - x_2^2)$	0	$3x_1^2(1 - x_2^2)$	$-x_1x_2^3$	0
$\mathbf{S}'_4$	0	$-x_2(1 - x_2^2)x_1^2$	0	$-x_1x_2(1 - x_2^2)$	$x_1^2(3x_2^2 - 1)$
$\mathbf{S}'_5$	$x_1^3x_2^2(1 - x_2^2)$	0	$3x_1^2x_2^2(1 - x_2^2)$	$x_1^3x_2(1 - 2x_2^2)$	0
$\mathbf{S}^*$	$u_1^* = 0$	$u_2^* = -x_2$	$e_{11}^* = 0$	$e_{12}^* = 0$	$e_{22}^* = -1$

State	$E'_{11} = \dfrac{3}{4}(2e'_{11} + e'_{22})$	$E'_{12} = \dfrac{3}{4}e'_{12}$	$E'_{22} = \dfrac{3}{4}(2e'_{22} + e'_{11})$
$\mathbf{S}'_1$	$\dfrac{3}{2}(1 - x_2^2)$	$-\dfrac{3}{4}x_1x_2$	$\dfrac{3}{4}(1 - x_2^2)$
$\mathbf{S}'_2$	$\dfrac{3}{4}(3x_2^2 - 1)$	0	$\dfrac{3}{2}(3x_2^2 - 1)$
$\mathbf{S}'_3$	$\dfrac{9}{2}x_1^2(1 - x_2^2)$	$-\dfrac{3}{4}x_1x_2^3$	$\dfrac{9}{4}x_1^2(1 - x_2^2)$
$\mathbf{S}'_4$	$\dfrac{3}{4}x_1^2(3x_2^2 - 1)$	$-\dfrac{3}{4}x_1x_2(1 - x_2^2)$	$\dfrac{3}{2}x_1^2(3x_2^2 - 1)$
$\mathbf{S}'_5$	$\dfrac{9}{2}x_1^2x_2^2(1 - x_2^2)$	$\dfrac{3}{4}x_1^3x_2(1 - 2x_2^2)$	$\dfrac{9}{4}x_1^2x_2^2(1 - x_2^2)$
$\mathbf{S}^*$	$E_{11}^* = -\dfrac{3}{4}$	$E_{12}^* = 0$	$E_{22}^* = -\dfrac{3}{2}$

TABLE 2: The complementary states.

State	ψ	$E''_{11} = \frac{\partial^2 \psi}{\partial x_2^2}$	$E''_{12} = -\frac{\partial^2 \psi}{\partial x_1 \, \partial x_2}$	$E''_{22} = \frac{\partial^2 \psi}{\partial x_1^2}$
S''_1	$-\frac{1}{2} x_1^2$	0	0	-1
S''_2	$-\frac{1}{12} x_1^4$	0	0	$-x_1^2$
S''_3	$\frac{1}{2} x_2^2(1 - x_1^2)^2$	$(1 - x_1^2)^2$	$4x_1x_2(1 - x_1^2)$	$-2x_2^2(1 - 3x_1^2)$
S''_4	$\frac{1}{4} x_2^4(1 - x_1^2)^2$	$3x_2^2(1 - x_1^2)^2$	$4x_1x_2^3(1 - x_1^2)$	$-x_2^4(1 - 3x_1^2)$
S''_5	$\frac{1}{2} x_1^2x_2^2(1 - x_1^2)^2$	$x_1^2(1 - x_1^2)^2$	$-2x_2(3x_1^5 - 4x_1^3 + x_1)$	$x_2^2(15x_1^4 - 12x_1^2 + 1)$

State	$e''_{11} = \frac{4}{9}[2E''_{11} - E''_{22}]$	$e''_{12} = \frac{4}{3} E''_{12}$	$e''_{22} = \frac{4}{9}[2E''_{22} - E''_{11}]$
S''_1	$\frac{4}{9}$	0	$-\frac{8}{9}$
S''_2	$\frac{4}{9} x_1^2$	0	$-\frac{8}{9} x_1^2$
S''_3	$\frac{8}{9}[(1 - x_1^2)^2 + x_2^2(1 - 3x_1^2)]$	$\frac{16}{3} x_1x_2(1 - x_1^2)$	$-\frac{4}{9}[4x_2^2(1 - 3x_1^2) + (1 - x_1^2)^2]$
S''_4	$\frac{4}{9}[6x_2^2(1 - x_1^2)^2 + x_2^4(1 - 3x_1^2)]$	$\frac{16}{3} x_1x_2^3(1 - x_1^2)$	$-\frac{4}{9}[2x_2^4(1 - 3x_1^2) + 3x_2^2(1 - x_1^2)^2]$
S''_5	$\frac{4}{9}[2x_1^2(1 - x_1^2)^2 - x_2^2(15x_1^4 - 12x_1^2 + 1)]$	$-\frac{8}{3} x_2(3x_1^5 - 4x_1^3 + x_1)$	$\frac{4}{9}[2x_2^2(15x_1^4 - 12x_1^2 + 1) - x_1^2(1 - x_1^2)^2]$

TABLE 3: Values of $S'_i \cdot S'_j$ and $S^* \cdot S'_j$.

	S'_1	S'_2	S'_3	S'_4	S'_5	S^*
S'_1	3.866667	-0.8	3.6	0	0.537143	-2
S'_2	-0.8	4.8	-0.8	1.6	0.114286	0
S'_3	3.6	-0.8	6.045714	$-.365714$	0.925714	$-2.$
S'_4	0	1.6	$-.365714$	1.112381	0.045714	0
S'_5	0.537143	0.114286	0.925714	0.045714	.364080	-0.4
S^*	-2	0	-2	0	-0.4	6.

TABLE 4: Values of $r'_{ip} = a_{ip}/a_{pp}$ and of a^2_{pp}.

	$p=1$	2	3	4	5	a^2_{pp}
$i=1$	1.	0.206897	−0.928571	−0.190946	0.007212	0.258621
2	—	1.	0.011905	−0.343706	−0.035183	0.215774
3	—	—	1.	0.128711	−0.164771	0.371287
4	—	—	—	1.	−0.044662	1.940316
5	—	—	—	—	1.	4.776450

TABLE 5: Values of $\mathbf{S}''_i \cdot \mathbf{S}''_j$ and $\mathbf{S}^* \cdot \mathbf{S}''_j$.

	$\mathbf{S}''_1$	$\mathbf{S}''_2$	$\mathbf{S}''_3$	$\mathbf{S}''_4$	$\mathbf{S}''_5$	$\mathbf{S}^*$
$\mathbf{S}''_1$	3.555556	1.185185	0.948148	0.948148	0.135450	4.
$\mathbf{S}''_2$	1.185185	0.711111	−0.496649	−0.054180	−0.045150	1.333333
$\mathbf{S}''_3$	0.948148	−0.496649	8.777141	5.616633	0.456424	0
$\mathbf{S}''_4$	0.948148	−0.054180	5.616633	5.238667	0.247434	0
$\mathbf{S}''_5$	0.135450	−0.045150	0.456424	0.247434	1.465237	0
$\mathbf{S}^*$	4.	1.333333	0	0	0	6

TABLE 6: Values of $r''_{ip} = b_{ip}/b_{pp}$ and of b^2_{pp}.

	$p=1$	2	3	4	5	b^2_{pp}
$i=1$	1	−0.333333	−1.123812	0.113388	−0.098739	0.281250
2	—	1.	2.571430	−0.591654	0.201460	3.164056
3	—	—	1.	−0.685644	−0.039757	0.155412
4	—	—	—	1.	0.015344	0.654785
5	—	—	—	—	1.	0.700065

Writing

$$\mathbf{I}'_p = \sum_{i=1}^{p} a_{ip}\,\mathbf{S}'_i\,, \qquad p = 1, \cdots, 5 \tag{4.2}$$

where the constants a_{ip} are to be determined according to the usual orthogonalization scheme, we have for U_m in (3.5)

$$U_m = \frac{1}{Ea^2}\left[S^{*2} - \sum_{p=1}^{m}\left(\mathbf{S}^* \cdot \sum_{i=1}^{p} a_{ip}\,\mathbf{S}'_i\right)^2\right], \qquad m = 1, \cdots, 5. \tag{4.3}$$

In expanding, only products $a_{ip}a_{ip}$ ($i, j = 1 \cdots, p$ where $p = 1, \cdots, 5$) of the co-

efficients enter. Defining $r'_{ip} = a_{ip}/a_{pp}$, $(i = 1, \cdots, p$ for a fixed $p = 1, \cdots, 5)$ we have

$$a_{ip}a_{jp} = r'_{ip}r'_{jp}a^2_{pp}$$

where the r'_{ip} are obtained for each p as solutions of the $(p - 1)$ linear equations

$$\sum_{i=1}^{p-1} (\mathbf{S}'_j \cdot \mathbf{S}'_i) r'_{ip} = -\mathbf{S}'_j \cdot \mathbf{S}'_p, \qquad j = 1, \cdots, p - 1$$

whose coefficients are the products $\mathbf{S}'_i \cdot \mathbf{S}'_j$ of Table 3. The values of the r'_{ip} are given in Table 4 as well as the values of the quantities $a^2_{11}, a^2_{22}, \cdots, a^2_{55}$. From these values and the values of the $\mathbf{S}^* \cdot \mathbf{S}'_j$ in Table 3 we may compute $U_1, \cdots, U_5$ from (4.3).

Similarly, Table 5 gives the values of the scalar products of the complementary states with one another and with $\mathbf{S}^*$. Writing

$$\mathbf{I}''_q = \sum_{j=1}^{q} b_{jq}\mathbf{S}''_j$$

we get for (3.5)

$$L_n = \frac{1}{Ea^2} \sum_{q=1}^{n} \left(\mathbf{S}^* \cdot \sum_{j=1}^{q} b_{jq}\mathbf{S}''_j \right)^2, \qquad n = 1, \cdots, 5. \tag{4.4}$$

The products of the coefficients are given by

$$b_{iq}b_{jq} = r''_{iq}r''_{jq}b^2_{qq}, \qquad i, j = 1, \cdots, q \qquad q = 1, \cdots, 5 \tag{4.5}$$

where the values of the r''_{iq} and the b^2_{qq} are to be found in Table 6.

Evaluating (4.3) and (4.4) we find (reentering the factors E and a so that the quantity Ea^2 cancels out) the values of $L_1, \cdots, L_5$ and $U_1, \cdots, U_5$ given in Table 7 below.

TABLE 7

	L_n	U_m	
$m = n = 1$	4.500000	4.965516	(5 places accurate, 6'th estimated)
2	4.500000	4.928572	
3	4.676823	4.920992	
4	4.750435	4.890932	
5	4.761603	4.856468	(4 places accurate)

Entering these values in (3.6) we get successively improved inequalities for E as follows

TABLE 8

Iteration No.	
1	$.402778\,\frac{F}{a} \leq E \leq .444444\,\frac{F}{a}$
2	$.405797\,\frac{F}{a} \leq E \leq .444444\,\frac{F}{a}$
3	$.406422\,\frac{F}{a} \leq E \leq .427641\,\frac{F}{a}$
4	$.408920\,\frac{F}{a} \leq E \leq .421014\,\frac{F}{a}$
5	$.411822\,\frac{F}{a} \leq E \leq .420027\,\frac{F}{a}$

At the first iteration the gap between upper and lower bound is 10.3%. In four further iterations this is reduced to 1.99%. Averaging the upper and lower bounds yielded at the fifth iteration and retaining only 4 places gives the approximate formula

$$E = .4159\,\frac{F}{a} \tag{4.6}$$

where we have established that *the coefficient is in error by less than 1%*. The corresponding expression in terms of average stress and strain is

$$E = .8318\,\frac{\sigma}{\epsilon}. \tag{4.7}$$

It is interesting to compare these formulas with those for a test in which motion of the bar in the x_1-direction is not hindered, i.e. with the case of *pure compression in plane strain*. Labeling the uniform stress and strain by $\bar{\sigma}$ and $\bar{\epsilon}$ respectively, we at once obtain from Hooke's laws, written for plane strain, the relation

$$E = (1 - \nu^2)\,\frac{\bar{\sigma}}{\bar{\epsilon}}. \tag{4.8}$$

In the case $\nu = \frac{1}{3}$ which we have taken, this gives

$$E = .888889\,\frac{\bar{\sigma}}{\bar{\epsilon}}, \tag{4.9}$$

or

$$E = .444444\,\frac{\bar{F}}{\bar{a}} \tag{4.10}$$

where $\bar{F}$ is the load required to produce the compression $2\bar{a}$ in the simple compression test. Comparing these formulas with (4.6) and (4.7) we verify the fact that preventing slippage in the x_1-direction has the effect of stiffening the bar.

Given these formulas we can correct the value of E determined from a test in which it is assumed that the bar is in simple compression whereas actually due to frictional end effects the bar is in a state of compression akin to that of Fig. 2. Using (4.10) one arrives at a value $\overline{E}$ for E which is larger than the true value as given by (4.6). Division of these equations gives as the relation between E and $\overline{E}$

$$E = .9359\,\overline{E}, \tag{4.11}$$

so that .9359 is the correction factor to be used.

It is to be remarked that in practice one can not achieve even an approximate state of plane strain in compression unless a very long (in the x_3 direction) specimen is used. However, analogous results to those given here can be obtained for a specimen with *rotational symmetry*. The formulas deduced for this case would be of more immediate practical value in compression testing of materials.

ON THE NATURE OF THE BOUNDARY LAYER NEAR THE LEADING EDGE OF A FLAT PLATE*

BY

G. F. CARRIER (*Brown University*) AND C. C. LIN (*Massachusetts Institute of Technology*)

1. Introduction. The customary treatment of boundary layer flow[1] furnishes what is essentially an asymptotic expansion of the stream function (or an equivalent function defining the velocities). This solution is valid, of course, only in a region far downstream of the leading edge of the obstacle. Since one knows, however, that to all practical purposes a potential flow exists sufficiently far from the obstacle, there remains only a closed region in the neighborhood of the origin in which the nature of the flow is not known. It is the purpose of the present paper to show the nature of the flow of an incompressible fluid in this region when the obstacle is a flat plate.

2. General development. The equations governing an incompressible, viscous, fluid flow are, in conventional notation[2],

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial y_1} + \left(\frac{1}{\rho}\right) \frac{\partial p_1}{\partial x_1} = \nu \Delta u_1 \,, \tag{1}$$

$$u_1 \frac{\partial v_1}{\partial x_1} + v_1 \frac{\partial v_1}{\partial y_1} + \left(\frac{1}{\rho}\right) \frac{\partial p_1}{\partial y_1} = \nu \Delta v_1 \,, \tag{2}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0. \tag{3}$$

These may be reduced to a single equation in one unknown function in any of several ways. For convenience, we first introduce the following changes of variable: $p = p_1/\rho u_0^2$, $x = x_1 u_0/\nu$, $y = y_1 u_0/\nu$, $u = u_1/u_0 = \partial\psi/\partial y$, $v = v_1/u_0 = -\partial\psi/\partial x$, $x + iy = z = re^{i\theta}$. Here, u_0 is the free stream velocity. With the introduction of these variables, Eqs. (1), (2), (3), are readily reduced to the form

$$\partial^4\psi/\partial z^2\,\partial\bar{z}^2 = \frac{1}{2i}\left[\partial\psi/\partial\bar{z}\;\partial^3\psi/\partial z^2\,\partial\bar{z} - \partial\psi/\partial z\;\partial^3\psi/\partial z\,\partial\bar{z}^2\right]. \tag{4}$$

Here $\bar{z}$ is the complex conjugate of z and, we might note now, $u + iv = -2i\partial\psi/\partial\bar{z}$.

If we restrict our analysis to the flow past a flat plate such that as $x \to -\infty$, $\psi \to y$, we must impose the boundary conditions $\partial\psi/\partial\bar{z} \equiv 0$ when $\theta = 0$ and when $\theta = 2\pi$. Furthermore, we must anticipate that the line $\theta = 0$ will be a branch line of our solution since the higher derivatives of ψ will be discontinuous along this line. We may construct such a function which is a solution of Eq. (4) in the following manner. We note that

$$\begin{aligned}\psi_0 &= A[\bar{z}\,z^{1/2} + z\,\bar{z}^{1/2} - z^{3/2} - \bar{z}^{3/2}] \\ &= Ar^{3/2}\left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right)\end{aligned} \tag{5}$$

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[1] See, for instance, L. Prandtl, *The mechanics of viscous fluids*, in vol. 3 of W. F. Durand's *Aerodynamic theory*, p. 34, Springer, Berlin, 1934.

[2] The subscript unity is adopted so that we may use the variable without subscripts for the quantities to be introduced later.

is a biharmonic function for which $\theta = 0$ is a branch line, along which the velocity $\partial\psi/\partial\bar{z}$ vanishes. If we write Eq. (4) in the form

$$L(\psi) = L^*(\psi), \tag{6}$$

where L is the biharmonic operator and $L^*(\psi)$ is the rest of Eq. (4), then we may define

$$\begin{aligned} &L(\psi_0) = 0, \\ &L(\psi_1) = L^*(\psi_0), \\ &L(\psi_n) = L^*(\psi_0 + \cdots + \psi_{n-1}) - L^*(\psi_0 + \cdots + \psi_{n-2}), \\ &\cdots . \end{aligned} \tag{7}$$

Over any region in which the series

$$\psi = \psi_0 + \psi_1 + \cdots \tag{8}$$

converges then, we may take ψ (as defined above) as a solution to Eq. (4).[3] In this case, a particular integral for ψ_1 is

$$\psi_1^{(1)} = A^2 r^3[(\sin 2\theta)/40 - (\theta \cos \theta/64], \tag{9}$$

and a biharmonic function which when added to $\psi_1^{(1)}$ adjusts the conditions at the plate, is

$$\begin{aligned} \psi_1^{(2)} = \frac{A^2}{128} r^3[&\ln r(3 \sin \theta - \sin 3\theta) + \theta(3 \cos \theta - \cos 3\theta) \\ &- \frac{A^2 r^3}{20} (\alpha \sin \theta + \beta \sin 3\theta)], \end{aligned} \tag{10}$$

where $\alpha + 3\beta = 1$. As we shall see later, $\psi_1^{(2)} \ll \psi_0$ in the matching region so that the choice $\alpha = 1$ serves our purpose. The function ψ_1 becomes then

$$\begin{aligned} \psi_1 = \psi_1^{(1)} + \psi_1^{(2)} = [A^2 r^3/128][&\theta(\cos \theta - \cos 3\theta) + \ln r(3 \sin \theta - \sin 3\theta)] \\ &+ [A^2 r^3/40][\sin 2\theta - 2 \sin \theta]. \end{aligned} \tag{11}$$

It is fairly evident that the higher approximations will contain terms in $r^{9/2}$, r^{12}, $r^{9/2} \ln r$, $\cdots$, and will also involve coefficients with higher powers of A. As we (again) shall see later, $A = .083$. Thus, it is evident that within some circle this development is convergent and represents *some* flow. It is not immediately evident that the expansion for ψ converges for $y \ll 1$, $x > 1$. However, since our solution is of the Stokes type, it is to be anticipated that the leading term of the expansion describes the flow in a sufficiently thin neighborhood of the plate. Since we do not need to establish such a region of convergence, however, we shall not pursue the question further.[4]

3. The matching procedure. The solution obtained by Blasius for the velocity profile in the downstream region is usually presented in the form

[3] This implies that we base the expansion on the Stokes type slow-flow solutions.

[4] We note, however, that the choice of α for Eq. (10) and the corresponding "arbitrary" coefficients in ψ_n, where $n > 1$, would be governed largely by this convergence consideration.

$$u = f'(\eta), \qquad v = \frac{1}{2(x)^{1/2}}\,[\eta f'(\eta) - f(\eta)], \tag{12}$$

where, in our notation, $\eta = y/x^{1/2}$. In Eqs. (12), $f(\eta)$ can be given by a power series of the form[1]

$$f = \alpha\eta^2/2! - \alpha^2\eta^5/2\cdot 5! + \cdots, \tag{13}$$

where $\alpha = .332$.

Thus, $u = \alpha\eta - \alpha^2\eta^4/2\cdot 4! = \alpha r^{1/2} \sin\theta/(\cos\theta)^{1/2}$

$$-\alpha^2 r^2 \sin^4\theta/48\cos^2\theta + \cdots. \tag{14}$$

That is

$$u \simeq \alpha r^{1/2}\theta - \alpha^2 r^2\theta^4/48 + \cdots \tag{15}$$

in the neighborhood of $\theta = 0$.

The solution of the preceding section implies that the velocity component u (i.e. Re $(-2i\partial\psi/\partial\bar{z})$ be given by

$$u = Ar^{1/2}(5\sin\theta/2 + \sin 3\theta/2) + \cdots$$

or

$$u \simeq 4Ar^{1/2}\theta + \gamma r^2\theta^4 + \delta r^2 \ln r\theta^4 + \cdots. \tag{16}$$

Two facts are evident. If these functions are to represent the same phenomenon in any overlapping regions where r is not large, then $4A \equiv \alpha$. Furthermore, we note that these two functions can never be identically the same since the $r^2 \ln r$ term is lost in the asymptotic representation. The latter fact is quite in order since *neither* solution has the pretense of being a complete solution. Therefore, we may establish validity of the leading term of this representation (i.e. ψ_0) if we can find a region in which both ψ_0 and f are valid approximations to the rigorous solution. That is, we must find a region in which $L(\psi) - L^*(\psi) \ll \partial^4\psi/\partial r^4$ (say). Our method of solution and its results indicate that $r \simeq 1$, $\theta \ll 1$ is such a region. But for θ sufficiently small and r of order unity, $x^{1/2}f(\eta)$ is almost precisely the same function as ψ_0 (see Eq. (15)) and hence is as valid a solution. This, of course, can be worked out by computing $L[x^{1/2}f(\eta)] - L^*[x^{1/2}f(\eta)]$, but this is really not necessary. Hence, we have that in the neighborhood of $r = 0$, Eq. (5) demonstrates the nature of the flow field provided $A = .083$. Figure 1 shows the four regions into which the flow field may be divided and the solution applicable to each. In region IV, it is doubtful if any reasonably simple analytic expression could be found for ψ. However, it is suggested that the flow field could be computed if one combined some intelligent "first guessing" (interpolation essentially) with the relaxation procedure and applied these to the region IV.

4. A modified Blasius solution. The alternative procedure[5] for developing the foregoing solution leads to a slightly modified boundary layer solution which establishes more clearly the overlap of the slow-flow solution (which includes the leading edge) and the boundary layer solution.

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[5]Actually, the two authors arrived at the solution independently by the two procedures described here, but felt that a joint presentation provided the most lucid account of the matter.

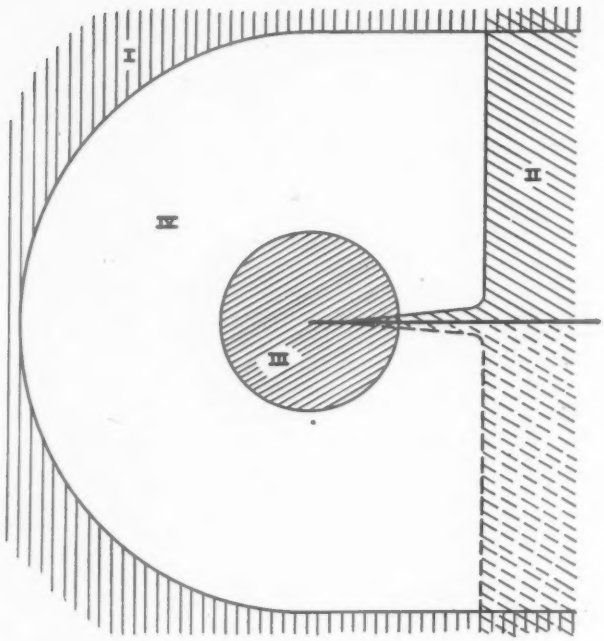

FIG. 1.

(I) Potential Solution (II) Blasius Solution (III) Slow Flow Solution

It seems that the most natural coordinate system to use for the flow past a flat plate is the parabolic coordinate system (ξ, η) defined by

$$x + iy = (\xi + i\eta)^2 \tag{17}$$

or

$$\xi + i\eta = r^{1/2} \exp(i\theta/2)$$

By using this system of coordinates, it is possible to give a unified discussion of the boundary layer solution and the slow motion solution in the neighborhood of the plate, including the leading edge.

In the parabolic system of coordinates, the equation for the stream function ψ becomes

$$L(\psi) = (\xi^2 + \eta^2)(\psi_\xi \Delta\psi_\eta - \psi_\eta \Delta\psi_\xi) - 2(\eta\psi_\xi - \xi\psi_\eta)\Delta\psi, \tag{18}$$

where

$$L(\psi) \equiv (\xi^2 + \eta^2)\Delta\Delta\psi - 4(\xi\Delta\psi_\xi + \eta\Delta\psi_\eta - \Delta\psi). \tag{19}$$

From a consideration of the diffusive effect of the viscous forces, it is obvious that the effect of viscosity is essentially limited to regions with $\eta \sim 1$, close to the solid boundary. For very small values of η, the motion is slow. Since $\psi = 0$ and $\partial\psi/\partial\eta = 0$ at $\eta = 0$ it is obvious that in this region a good approximation to (18) is

$$L(\psi) = 0. \tag{20}$$

A polynomial solution satisfying the boundary conditions $\psi = \psi_\eta = 0$ at $\eta = 0$ is

$$\psi = A\xi\eta^2 \tag{21}$$

which agrees with Eq. (5) of Sec. 2. This solution can also be obtained from Eq. (19) by writing $\psi = \eta^2(f(\xi) + 0(\eta))$ for small η.

If ξ is large while $\eta \sim 1$, one may introduce a new variable

$$\xi' = \epsilon\xi \tag{22}$$

where ϵ is a small parameter.[6] In the limit $\epsilon \to 0$, the equation (3) becomes

$$\xi'^2 \frac{\partial^4 \psi^{(0)}}{\partial \eta^4} = \xi'^2(\psi^{(0)}_{\xi'}\psi^{(0)}_{\eta\eta\eta} - \psi^{(0)}_{\eta}\psi^{(0)}_{\xi'\eta\eta}) + 2\xi'\psi^{(0)}_{\eta}\psi^{(0)}_{\eta\eta} \tag{23}$$

where

$$\psi^{(0)} = \epsilon\psi \tag{24}$$

This equation may be solved by putting

$$\psi^{(0)} = \xi' f(\eta); \tag{25}$$

then $f(\eta)$ satisfies

$$f^{iv} = ff''' + f'f'', \tag{26}$$

or

$$f''' = ff''; \tag{27}$$

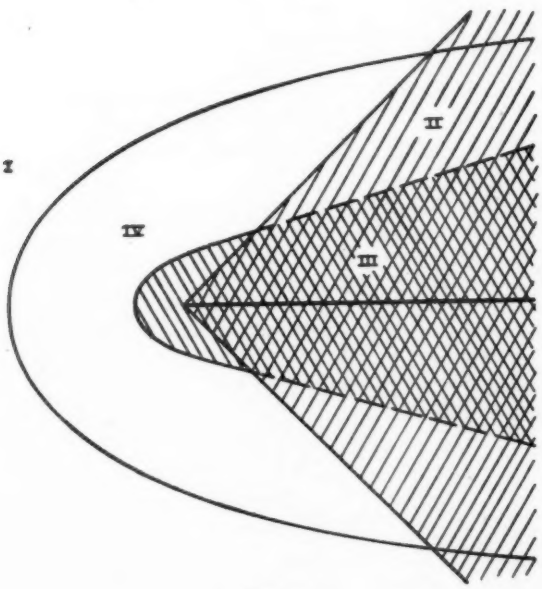

FIG. 2.

(I) Potential Solution (II) Modified Blasius Solution (III) Slow Flow Solution

[6] To give ϵ an explicit meaning, we consider the region $\eta \leq \epsilon\xi$.

this is the usual boundary layer equation except that η is not quite the conventional parameter.[7] Higher approximations may be obtained by solving (18) in powers of ϵ.

Thus, the regions of approximate validity of these solutions are as follows:

(*i*) potential solution: $\eta \gg 1$;

(*ii*) boundary layer equation: $\eta \sim 1$, $\eta \ll \xi$;

(*iii*) Stokes equation: $\eta \ll 1$, irrespective of ξ.

These regions are indicated in Fig. 2. It is obvious that there is an overlap of regions (*ii*) and (*iii*) where the solution (5) agrees with the boundary layer solution. Again, we have not established the convergence of the slow flow solution. However, the physical considerations which lead to the slow flow and the boundary layer solutions indicate that they are at least valid asymptotic approximations, if not convergent processes. This is further strengthened by the identical nature of the leading terms of the two types of solutions for $\eta \ll 1$.

[7] Here, $\eta = r^{1/2} \sin \theta/2$, conventionally $\eta = r^{1/2} \sin \theta/(\cos \theta)^{1/2}$.

THE APPROXIMATE NUMERICAL SOLUTION OF THE NON-HOMOGENEOUS LINEAR FREDHOLM INTEGRAL EQUATION BY RELAXATION METHODS*

BY

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Summary. Relaxation methods are applied to the problem of finding an approximate numerical solution to the non-homogeneous linear Fredholm integral equation. As an illustration of the technique the deflection of a simply supported single span beam subjected to both normal and end loads is found.

1. The approximation to the integral equation. Consider the equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x, t)\varphi(t)\, dt, \tag{1}$$

$\varphi(x)$ being the unknown function.

Let the interval $[a, b]$, of length L, be subdivided equally into n parts as in Fig. 1., so that each subinterval is of length $L/n = l$ say.

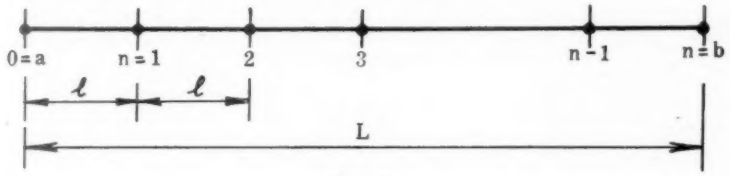

FIG. 1.

Then equation (1) may be replaced by the approximation

$$\varphi(h) = f(h) + \lambda l\left[\frac{1}{2}\{K(h, a)\varphi(a) + K(h, b)\varphi(b)\} + \sum_{j=1}^{n-1} K(h, j)\varphi(j)\right], \tag{2}$$

where $\varphi(h)$ represents the value of $\varphi(x)$ at the point $x = h$, $h = a, 1, 2, \cdots, b$, and the integral $\int_h^{h+1} A(x)\, dx$ has been replaced by the approximation $\frac{1}{2}l(A(h) + A(h + 1))$. To obtain an approximate solution to equation (1) it is necessary, then, to solve the set of $n + 1$ simultaneous equations (2). This may conveniently be done by using the Relaxation technique.**

2. The residual and the relaxation operators. As a preliminary to solving, approximately, equations (2) it is necessary to construct two operators which, in the language of relaxation methods, are known as the residual operator and the relaxation operator respectively.

Let it be supposed that by means of an inspired guess (or by some other means) approximate values have been obtained for all the $\varphi(h)$. In general, the values will not be correct, i.e. will not satisfy equations (2), so that the error at each point is given by

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**Some knowledge of the technique is assumed, and it will not be discussed here. For further information see References 1, 2.

$$F(h) = f(h) + \lambda l\left[\frac{1}{2}\{K(h, a)\varphi(a) + K(h, b)\varphi(b)\} + \sum_{j=1}^{n-1} K(h, j)\varphi(j)\right] - \varphi(h), \quad (3)$$

$F(h)$ being known as the residual at the point h, $h = a, 1, \cdots, n-1, b$.

Obviously, for an approximate solution of equations (2) we require that each $F(h)$ be (approximately) zero. To achieve this use is made of a relaxation operator.

Consider the effect of making an alteration of amount $\Delta\varphi(3)$ to the value of $\varphi(3)$ say. The resultant change in the residual at point 3, i.e. $F(3)$, is given by

$$\Delta F(3) = \frac{\partial F(3)}{\partial \varphi(3)} \Delta\varphi(3),$$

$$= [\lambda l K(3, 3) - 1]\Delta\varphi(3)$$

from equation (3). At the same time, and as a consequence of $\Delta\varphi(3)$, there is also a change in $F(2)$ of amount

$$\Delta F(2) = \frac{\partial F(2)}{\partial \varphi(3)} \Delta\varphi(3)$$

$$= \lambda l K(2, 3)\Delta\varphi(3),$$

with similar changes taking place in the residuals at the other h stations.

More generally, the following effects result:

$$\left.\begin{aligned}
\Delta F(h, j) &= [\lambda l K(h, j) - \delta_{hj}]\Delta\varphi(j) \quad , \\
\Delta F(h, a) &= \tfrac{1}{2}\lambda l K(h, a)\Delta\varphi(a) \quad , \\
\Delta F(h, b) &= \tfrac{1}{2}\lambda l K(h, b)\Delta\varphi(b) \quad , \\
\Delta F(a, a) &= [\tfrac{1}{2}\lambda l K(a, a) - 1]\Delta\varphi(a) \quad , \\
\Delta F(a, h) &= \lambda l K(a, h)\Delta\varphi(h) \quad , \\
\Delta F(a, b) &= \tfrac{1}{2}\lambda l K(a, b)\Delta\varphi(b) \quad , \\
\Delta F(b, b) &= [\tfrac{1}{2}\lambda l K(b, b) - 1]\Delta\varphi(b) \quad , \\
\Delta F(b, h) &= \lambda l K(b, h)\Delta\varphi(h) \quad , \\
\Delta F(b, a) &= \tfrac{1}{2}\lambda l K(b, a)\Delta\varphi(a) \quad ,
\end{aligned}\right\} \quad h, j = 1, 2, \cdots, n-1, \quad (4)$$

where $\Delta F(h, j)$ is the change in the residual at station h resulting from a change in the value of φ, of amount $\Delta\varphi$, at station j. δ_{hj} is the kronecker delta*. In effect expressions (4) may be regarded as operators, and are known as the relaxation operators. For any arbitrary alteration to the value of the unknown function φ at any one station

*$\delta_{hj} = \begin{cases} 1, \text{ if } h = j, \\ 0, \text{ if } h \neq j, \end{cases}$

they give the resulting alterations to the errors in the solution at all stations. Thus they provide a means whereby the errors, or residuals, may be systematically reduced to as small a value as desired.

For actual use these operators are evaluated in tabular form. On putting $\Delta\varphi(h) = 1$, and dropping the Δ from $\Delta F(h, j)$ we have the unit operations Table I.

TABLE 1. RELAXATION OPERATORS

$\Delta\varphi(h) = 1$, all the rest zero.	Alteration to Residual $F(J)$.					
	$F(a)$	$F(1)$	$F(2)$	$\cdots$	$F(n-1)$	$F(b)$
$h = a$	$\frac{1}{2}\lambda lK(a, a) - 1$	$\frac{1}{2}\lambda lK(1, a)$	$\frac{1}{2}\lambda lK(2, a)$	$\cdots$	$\frac{1}{2}\lambda lK(n-1, a)$	$\frac{1}{2}\lambda lK(b, a)$
$h = 1$	$\lambda lK(a, 1)$	$\lambda lK(1, 1) - 1$	$\lambda lK(2, 1)$	$\cdots$	$\lambda lK(n-1, 1)$	$\lambda lK(b, 1)$
$h = 2$	$\lambda lK(a, 2)$	$\lambda lK(1, 2)$	$\lambda lK(2, 2) - 1$	$\cdots$	$\lambda lK(n-1, 2)$	$\lambda lK(b, 2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$h = n-1$	$\lambda lK(a, n-1)$	$\lambda lK(1, n-1)$	$\lambda lK(2, n-1)$	$\cdots$	$\lambda lK(n-1, n-1) - 1$	$\lambda lK(b, n-1)$
$h = b$	$\lambda lK(a, b)$	$\frac{1}{2}\lambda lK(1, b)$	$\frac{1}{2}\lambda lK(2, b)$	$\cdots$	$\frac{1}{2}\lambda lK(n-1, b)$	$\frac{1}{2}\lambda lK(b, b) - 1$

3. An example. In order to illustrate the method an example will be worked. Consider a single span beam simply supported at the ends and subjected to both normal and end loads. Both the normal loads and the moment of inertia of the beam may vary in any manner whatever. It is required to calculate the deflection of the beam.

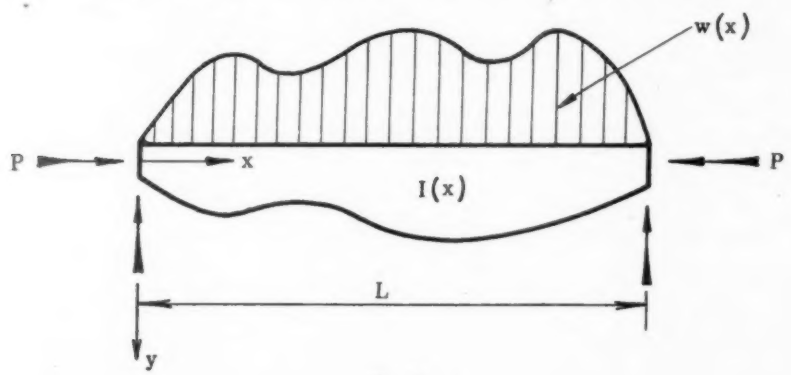

FIG. 2.

With the notation shown in Fig. 2., and using the Bernoulli-Euler theory of bending, the governing differential equation* is

*Fig. 2 is diagrammatic only. Actually it is assumed that the neutral plane of the beam is straight, that the beam cross section is symmetrical about a vertical axis, and that the force P acts collinearly with the resulting straight line axis.

$$\frac{d^2y}{dx^2} = -\frac{M(x)}{E\,I(x)}$$

which, for the simply supported beam shown, becomes

$$\frac{d^2y}{dx^2} = -\frac{1}{E\,I(x)}\,(Py + M_1(x)). \tag{5}$$

$M_1(x)$ is the bending moment associated with the normal loading system $w(x)$, and so for any given $w(x)$ may be considered as known. $I(x)$ is the relevant moment of inertia of the beam cross section and may also be considered as known.

The integral equation appropriate to the problem is

$$y(x) = \int_0^L K(x, \xi)A(\xi)M_1(\xi)\,d\xi + \int_0^L K(x, \xi)PA(\xi)y(\xi)\,d\xi, \tag{6}$$

where $A(\xi) = 1/E\,I(\xi)$,

$$K(x, \xi) = \begin{cases} \xi(1 - x/L) \text{ for } 0 \leq \xi \leq x, \\ x(1 - \xi/L) \text{ for } x \leq \xi \leq L, \end{cases}$$

and $M_1(x)$ can be evaluated for each point x once $w(x)$ is specified.

The integral equation is, therefore, of the form

$$y(x) = f(x) + \lambda \int_0^L K(x, \xi)y(\xi)\,d\xi$$

where $f(x)$, λ may be considered as known.

For comparison of accuracy it is convenient to choose a particular problem for which an exact solution is known. Accordingly it will be assumed that the load $w(x)$ is a uniformly distributed load, and that the moment of inertia is constant. For this particular case equation (6) becomes

$$y(x) = \frac{w}{24E\,I}\,(x^4 + L^3x - 2Lx^3) + \frac{P}{E\,I}\int_0^L K(x, \xi)y(\xi)\,d\xi, \tag{7}$$

$K(x, \xi)$ being as given earlier.

As with most numerical procedures it is desirable to make the quantities concerned non-dimensional. To do this, put

$$x = x_1L, \qquad y = y_1L, \qquad PL^2/EI = A, \qquad wL^3/EI = B.$$

On dropping the subscript 1 (for simplicity) equation (7) becomes

$$y(x) = B\,\frac{x}{24}\,(x^3 + 1 - 2x^2) + A\int_0^1 K(x, \xi)y(\xi)\,d\xi, \qquad 0 \leq x \leq 1. \tag{8}$$

For convenience* we choose $\lambda = A = 2.5$, $B = 0.5$, and take $n = 10$ so that $l = 1/10$. The relaxation operator table may then be completed without further discussion, the result being Table II.

*The exact value of the deflection at the center of the span is given by

$$\frac{y}{L} = \frac{5}{384}\,\frac{w\,L^3}{EI}\,\frac{24(\sec u - 1 - \tfrac{1}{2}u^2)}{5u^4},$$

where $u = (L/2)(P/EI)^{1/2}$. As $u \to 0$, $\sec u \to 1$, as $u \to \pi/2$, $\sec u \to \infty$. Thus for $u = \pi/4$ say, $PL^2/EI = A = \pi^2/4 = 2.5$. Also, with $y/L = (5/384)wL^3/EI = 1/150$, $B = 1/2$.

TABLE No. II

RELAXING OPERATORS

$\Delta y(h) = 1$	$F(a)$	$F(1)$	$F(2)$	$F(3)$	$F(4)$	$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(b)$
$h = a$	−1.0000	0	0	0	0	0	0	0	0	0	0
1	0	−0.9775	0.0200	0.0175	0.0150	0.0125	0.0100	0.0075	0.0050	0.0025	0
2	0	0.0200	−0.9600	0.0350	0.0300	0.0250	0.0200	0.0150	0.0100	0.0050	0
3	0	0.0175	0.0350	−0.9475	0.0450	0.0375	0.0300	0.0225	0.0150	0.0075	0
4	0	0.0150	0.0300	0.0450	−0.9400	0.0500	0.0400	0.0300	0.0200	0.0100	0
5	0	0.0125	0.0250	0.0375	0.0500	−0.9375	0.0500	0.0375	0.0250	0.0125	0
6	0	0.0100	0.0200	0.0300	0.0400	0.0500	−0.9400	0.0450	0.0300	0.0150	0
7	0	0.0075	0.0150	0.0225	0.0300	0.0375	0.0450	−0.9475	0.0350	0.0175	0
8	0	0.0050	0.0100	0.0150	0.0200	0.0250	0.0300	0.0350	−0.9600	0.0200	0
9	0	0.0025	0.0050	0.0075	0.0100	0.0125	0.0150	0.0175	0.0200	−0.9775	0
b	0	0	0	0	0	0	0	0	0	0	−1.0000

To compute the initial residuals, note that if the initial solution is taken to be $y = 0$ everywhere, then $F(h) = f(h)$, $h = a, 1, \cdots, 9, b$. This gives the first line in the relaxation table which follows—Table III. The relaxation process is then carried out in the usual manner. It will be seen that, since the operators are extremely powerful, convergence is very rapid.

From the nature of the problem solved here, and by inspection of the relaxation table, it is obvious that advantage could be taken of symmetry to cut down the work. It would merely have entailed combining operators 1 and 9, 2 and 8, etc., in Table II, and then only relaxing the residuals $F(a)$ to $F(5)$. For clarity however it was thought advisable not to do this.

4. Accuracy of solution. The exact value of the deflection is given by[3]

$$\frac{y}{L} = \frac{w L^3}{16 EI u^4}\left[\frac{\cos u(1 - 2x/L)}{\cos u} - 1\right] - \frac{w L}{8 EI u^2} x(L - x),$$

where $u = (L/2)(P/EI)^{1/2}$.

Making this expression non-dimensional by use of the previous substitutions, and evaluating it for points $h = a, 1, \cdots, 5$, the following results are obtained:

	$h = a$	1	2	3	4	5 (center)
Exact Solution.	0	0.002,732	0.005,175	0.007,094	0.008,308	0.008,727
Approximate Solution.	0	0.002,737	0.005,185	0.007,106	0.008,331	0.008,751

The agreement is seen to be very good.

If greater accuracy were desired this could readily be obtained without increasing the number of subintervals by using a better approximation for the integration process. Thus, for example, Simpson's three point integration formula* could be used at the small expense of increasing slightly the initial labour necessary to compute the relaxation operators. However for some problems, depending on the terminal conditions, it might be desirable to decrease the lengths of the subintervals adjacent to the two ends a and b.

It is evident that the method as given here may be applied equally well to obtain an approximate solution to the non-homogeneous linear Volterra integral equation.

References

1. R. V. Southwell, *Relaxation methods in engineering science*, Oxford Univ. Press, 1940.
2. F. S. Shaw, *An introduction to relaxation methods (approximate methods of numerical computation)*, C. S. I. R. Div. of Aero. Report S M 78, Sept. 1946.
3. S. Timoshenko, *Theory of elastic stability*, McGraw-Hill Book Co., Inc., 1936. p. 8.
4. R. V. Southwell, *Relaxation methods in theoretical physics*, Oxford Univ. Press, 1946. p. 15.

*The use of more complicated integration formulae do not always lead to greater accuracy than that obtained by use of the simple trapezoidal rule. See, for example, ref. 4.

TABLE No. III

	Station	Displacements $\times 10^4$	Residuals $\times 10^7$										
			$F(a)$	$F(1)$	$F(2)$	$F(3)$	$F(4)$	$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(b)$
1		0 everywhere	0	20438	38667	52938	62000	65104	62000	52938	38667	20438	0
2	5	7.000	0	875	1750	2625	3500	−65625	3500	2625	1750	875	0
3			0	21313	40417	55563	65500	− 521	65500	55563	40417	21313	0
4	4	7.000	0	1050	2100	3150	−65800	3500	2800	2100	1400	700	0
5	6	7.000	0	700	1400	2100	2800	3500	−65800	3150	2100	1050	0
6			0	23063	43917	60813	2500	6479	2500	60813	43917	23063	0
7	3	6.000	0	1050	2100	−56850	2700	2250	1800	1350	900	450	0
8	7	6.000	0	450	900	1350	1800	2250	2700	−56850	2100	1050	0
9			0	24563	46917	5313	7000	10979	7000	5313	46917	24563	0
10	2	5.000	0	1000	−48000	1750	1500	1250	1000	750	500	250	0
11	8	5.000	0	250	500	750	1000	1250	1500	1750	−48000	1000	0
12			0	25813	− 583	7813	9500	13479	9500	7813	− 583	25813	0
13	1	3.000	0	−29325	600	525	450	375	300	225	150	75	0
14	9	3.000	0	75	150	225	300	375	450	525	600	−29325	0
15			0	− 3437	167	8563	10250	14220	10250	8563	167	− 3437	0
16	5	1.500	0	188	375	562	750	−14062	750	562	375	188	0
17			0	− 3249	542	9125	11000	158	11000	9125	542	− 3249	0
18	4	1.300	0	195	390	585	−12220	650	520	390	260	130	0
19	6	1.300	0	130	260	390	520	650	−12220	585	390	195	0
20			0	− 2924	1192	10100	− 700	1458	− 700	10100	1192	− 2924	0
21	3	1.100	0	193	385	−10423	495	413	330	247	165	83	0
22	7	1.100	0	83	165	247	330	413	495	−10423	385	193	0

TABLE No. III—*Continued*

	Station	Displacements $\times 10^4$	Residuals $\times 10^7$										
			$F(a)$	$F(1)$	$F(2)$	$F(3)$	$F(4)$	$F(5)$	$F(6)$	$F(7)$	$F(8)$	$F(9)$	$F(b)$
23			0	− 2648	1742	− 76	125	2284	125	− 76	1742	− 2648	0
24	5	0.200	0	25	50	75	100	− 1875	100	75	50	25	0
25			0	− 2623	1792	− 1	225	409	225	− 1	1792	− 2623	0
26	2	0.200	0	40	− 1920	70	60	50	40	30	20	10	0
27	8	0.200	0	10	20	30	40	50	60	70	− 1920	40	0
28			0	− 2573	− 108	99	325	509	325	99	− 108	− 2573	0
29	1	−0.300	0	2933	− 60	− 53	− 45	− 38	− 30	− 23	− 15	− 8	0
30	9	−0.300	0	− 8	− 15	− 23	− 30	− 38	− 45	− 53	− 60	2933	0
31			0	352	− 183	23	250	433	250	23	− 183	352	0
—	—	—	—	—	—	—	—	—	—	—	—	—	—
59	Final Residuals		0	1	0	−4	. 4	1	4	−4	0	1	0

Complete Displacements $y(h)$: $y(a) = y(b) = 0$; $y(1) = y(9) = 0.002{,}737$; $y(2) = y(8) = 0.005{,}185$; $y(3) = y(7) = 0.007{,}106$; $y(4) = y(6) = 0.008{,}331$; $y(5) = 0.008{,}751$

—NOTES—

ON BERNOULLI'S METHOD FOR SOLVING ALGEBRAIC EQUATIONS*

By BERNARD DIMSDALE (*Ballistic Research Laboratories, Aberdeen Proving Ground, Md.*)

1. Introduction. Both Graeffe's and Bernoulli's method for the solution of algebraic equations have the considerable advantage that no initial approximation to a root is required. Graeffe's method, although the convergence is quadratic, leads to factors[1] whose roots are large powers of the roots of the original equation, and for which the determination of angle is therefore quite difficult. Bernoulli's method, as amplified below, leads to factors of the original equation and therefore avoids this difficulty. While it is true that the convergence is slower (doubling the number of steps is roughly equivalent to one step of the Graeffe process) this is not a consideration of first importance if high speed computing machinery is available. Also, there is the possibility of applying rapidly convergent processes when a "sufficiently good" approximation is obtained. Furthermore, the method applies equally well to equations with complex coefficients.

Bernoulli's method consists of defining a sequence s_n in terms of the coefficients of the original equation, and taking quotients s_{n+1}/s_n . If these quotients converge, they converge to the root of largest modulus. If not, there are at least two roots of largest modulus. It will be seen that in this case successive systems of linear equations can be defined in terms of the s_n whose solutions will tend under certain conditions to the coefficients of a factor of the original equation.

2. Preliminary remarks. Let

$$p_N(z) = \sum_{\nu=0}^{N} a_\nu z^{N-\nu}, \qquad a_0 = 1 \tag{1}$$

with unequal roots $z_1 , z_2 , \cdots , z_{N'}$, of multiplicities $c_1 , c_2 , \cdots , c_{N'}$, respectively. The notation is chosen so that no modulus exceeds any preceding modulus. Let

$$P_M(z) = \sum_{\nu=0}^{M} \alpha_\nu z^{M-\nu}, \qquad \alpha_0 = 1, \qquad M \leq N' \tag{2}$$

with roots $z_1 , z_2 , \cdots , z_M$, and $M < N'$ if $z_{N'} = 0$. Let

$$s_n = \sum_{\nu=1}^{N'} c_\nu z_\nu^n , \qquad s_n' = \sum_{\nu=1}^{M} z_\nu^n$$

for all[2] n. We then have Newton's identities for s_n :

$$s_0 = N,$$

$$\sum_{\nu=0}^{n-1} a_\nu s_{n-\nu} + n a_n = 0, \qquad n = 1, \cdots , N-1,$$

$$\sum_{\nu=0}^{N} a_\nu s_{n-\nu} = 0, \qquad n \geq N.$$

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[1] Bodewig, E. *On Graeffe's method for solving algebraic equations*, Q. Appl. Math. **4**, 177-190 (1946). The methods described there apply only to equations with real coefficients.

[2] If $z_{N'} = 0$, then s_0 is not defined. Setting $s_0 = N$, as we are about to do, however, leads to no error in the rest of the sequence.

by which s_n can be calculated directly from the coefficients of (1) for all[3] n. The same identities hold for $s_{n'}$ when N, a, and s are replaced throughout by M, α, s'.

Consider the system of equations linear in $\alpha_{\nu n}$

$$\sum_{\nu=0}^{M} \alpha_{\nu n} s_{p-\nu} = 0, \qquad \alpha_{0n} = 1, \tag{3}$$

where $p = n, n+1, \cdots, n+M-1$; $n \geq M$. It will be shown in the next section that if $|z_M| > |z_{M+1}|$, then the system has a unique solution for all sufficiently large n and that $\alpha_{\nu n} - \alpha_\nu = O(z_{M+1}/z_M)^n$, so that $\alpha_{\nu n} \to \alpha_\nu$ as $n \to \infty$. Hence for all such values of M a factor of (1) in the form (2) is obtained, and (1) can be separated into factors each of which has roots of equal modulus. Then for each of these factors a shift of origin will result in further factorization.

3. Convergence. Let

$$\Delta_{0n}(s) = \begin{vmatrix} s_{n-1} & s_{n-2} & \cdots & s_{n-M} \\ s_n & s_{n-1} & \cdots & s_{n-M+1} \\ \vdots & \vdots & & \\ s_{n+M-2} & s_{n+M-3} & \cdots & s_{n-1} \end{vmatrix}$$

and let $\Delta_{\nu n}(s)$ for $\nu = 1, 2, \cdots, M$ be the same determinant with the column headed by $s_{n-\nu}$ replaced by $s_n, s_{n+1}, \cdots, s_{n+M-1}$, so that if $\Delta_{0n}(s) \neq 0$, then $\alpha_{\nu n} = -\Delta_{\nu n}(s)/\Delta_{0n}(s)$. Let

$$\delta_\nu(z_{\lambda 1}, z_{\lambda 2} \cdots, z_{\lambda M}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_{\lambda 1} & z_{\lambda 2} & \cdots & z_{\lambda M} \\ z_{\lambda 1}^M & z_{\lambda 2}^M & \cdots & z_{\lambda M}^M \end{vmatrix}, \qquad \nu = 0, 1, \cdots, M,$$

where the row with $z^{M-\nu}$ is absent. For brevity let $\delta_\nu = \delta_\nu(z_1, z_2, \cdots, z_M)$. We note that δ_0^2 is the discriminant of (2) and is therefore not zero, since all the roots of (2) are different. Then, by factoring,

$$\Delta_{\nu n}(s') = \pm(z_1, z_2 \cdots z_M)^{n-M} \delta_0 \delta_\nu \tag{4}$$

and, using the theorem for addition of determinants

$$\Delta_{\nu n}(s) = \pm \sum (c_{\lambda 1} c_{\lambda 2} \cdots c_{\lambda M})(z_{\lambda 1} \cdots z_{\lambda M})^{n-M} \delta_0(z_{\lambda 1}, \cdots, z_{\lambda M}) \delta_\nu(z_{\lambda 1}, \cdots, z_{\lambda M})$$

$$= (c_1 c_2 \cdots c_M)\Delta_{\nu n}(s') \pm \sum{}',$$

where $\sum$ denotes summation over all combinations of the M integers $\lambda_1 < \lambda_2 < \cdots < \lambda_M$ chosen from $1, 2, \cdots, N'$, and $\sum'$ is the same sum with $(1, 2, \cdots, M)$ omitted.

From (4), we have $\Delta_{0n}(s') \neq 0$ for all allowed values of M. Since the z's have been

[3] The original Bernoulli method permits an arbitrary selection of the first N of the s_n. In case there are two equal roots of largest modulus and the remaining roots have smaller moduli, it can be shown that for an arbitrary selection the convergence of s_{n+1}/s_n will in general be $O(1/n)$ whereas for this choice it will be in our present notation, $O(z_2/z_1)^n$.

arranged in order of non-increasing moduli, it follows that if $|z_{M+1}| < |z_M|$, then $\Delta_{0n}(s)/\Delta_{0n}(s') = (c_1c_2 \cdots c_M) + O(z_{M+1}/z_M)^n$, and therefore that for sufficiently large n, $\Delta_{0n}(s) \neq 0$. Hence, for such n, a unique solution of (3) exists. If δ_r is not zero, we can also write $\Delta_{rn}(s)/\Delta_{rn}(s') = (c_1c_2 \cdots c_M) + O(z_{M+1}/z_M)^n$ from which

$$\Delta_{rn}(s)/\Delta_{0n}(s) = [1 + O(z_{M+1}/z_M)^n]\Delta_{rn}(s')/\Delta_{0n}(s'),$$

that is

$$\alpha_{rn} = [1 + O(z_{M+1}/z_M)^n]\alpha_r .$$

Therefore, $\alpha_{rn} - \alpha_r = O(z_{M+1}/z_M)^n$. This equation is easily established if δ_r, that is, α_r, is zero.

It might be remarked finally that if $M > N'$, the system (3) is dependent for every n; if $M = N'$, then $\alpha_{rn} = \alpha_r$ for all n if $z_{N'} \neq 0$, and if $z_{N'} = 0$ then in this case (3) is again dependent for all n.

4. Example. We apply the method to the equation

$$z^5 - 3z^4 - (2 + i)z^3 + (12 + 5i)z^2 - (8 + 8i)z + 4i = 0,$$

whose roots are

$$z_1 = -1 - (1 + i)^{1/2} = -2.098684113 - 0.4550898608i,$$

$$z_2 = 2, \qquad z_3 = 1,$$

$$z_4 = -1 + (1 + i)^{1/2} = .0986841134 + .4550898608i,$$

and $c_1 = c_3 = c_4 = 1$, $c_2 = 2$. Here $|z_1| = 2.147459380$, $|z_4| = .4656665498$. We observe that $|z_2|/|z_1| = .931$, $|z_3|/|z_2| = .5$ and so we would expect the quotients to converge much more slowly than the solutions of the second order system. We observe that as a matter of fact convergence of the quotient is not visible for $n \leq 30$, but that $n = 30$ gives us nine place accuracy in the second order system.

On solving the quadratic

$$z^2 + \alpha_{1,29}z + \alpha_{2,29} = 0,$$

we obtain

$$z_1 = -2.098684113 - 0.455089863i, \qquad z_2 = 1.999999998 + 0.000000004i.$$

The third order system is not computed, because significant digits are lost in the computation. If we carry ten significant figures throughout, we will have only one or two left toward the end of our computation for $n = 30$. This corresponds to the fact that the second order system has converged to eight or nine significant figures by this time. This is true in general of the behaviour of these systems, and therefore one would, in practice, only carry the method to the lowest order which converges with sufficient rapidity ("sufficient" in this case referring to the speed with which these computations can be performed). On removal of the factor obtained, one could repeat the operation with the reduced equation. It is interesting to observe in our present example that since the quotients converge, we will, if we go far enough, begin to lose significant digits in the second order system, but that in the meantime we will have obtained all the accuracy possible with ten digits.

For any n, the fourth order system has the solution $\alpha_{1,n} = -1$, $\alpha_{2,n} = -4 - i$,

n	$R(s_n)$	$I(s_n)$	$R(s_n/s_{n-1})$	$I(s_n/s_{n-1})$	$R(\alpha_{1,n})$	$I(\alpha_{1,n})$	$R(\alpha_{2,n})$	$I(\alpha_{2,n})$
0	5	0						
1	3	0	0.60000	0				
2	13	2	4.3333	0.66667	0.007417	0.641533	−2.60445	−0.784920
3	9	−6	0.60694	−0.55491	0.384898	0.587371	−4.27631	−0.801871
4	47	16	2.7949	3.6410	0.1263745	0.4572780	−4.047214	−0.8663713
5	43	−40	0.56024	−1.0418	0.1404347	0.4418591	−4.142651	−0.8744711
6	157	94	0.86721	2.9928	0.1125302	0.4637507	−4.149824	−0.9158061
7	241	−210	0.54045	−1.6612	0.1095630	0.4520286	−4.182332	−0.8967605
8	451	448	0.14299	1.9835	0.1026775	0.4572601	−4.187307	−0.9110257
9	1359	−912	0.50564	−2.5244	0.1009175	0.4543819	−4.192759	−0.9071229
10	933	1762	−0.12656	1.2116	0.0998953•	0.4556626	−4.195222	−0.9103003
11	7241	−3190	0.28554	−3.9583	0.0991674	0.4549566	−4.196049	−0.9094951
12	143	5264	−0.25167	0.61610	0.0990177	0.4551808	−4.196893	−0.9101518
13	35675	−7384	−1.2177	−6.8102	0.09879357	0.45507276	−4.1970161	−0.91004092
14	−11075	6718	−0.33506	0.11896	0.09877110	0.45510405	−4.1972566	−0.91015585
15	160609	5854	−10.367	−6.8170	0.09871069	0.45508979	−4.1972786	−0.91015344
16	−65789	−55552	−0.42167	−0.33051	0.09870581	0.45509142	−4.1973402	−0.91016976
17	650015	206176	−7.3126	3.0408	0.09869098	0.45509074	−4.1973463	−0.91017495
18	−195899	−609214	−0.54393	−0.76470	0.09868935	0.45508986	−4.1973608	−0.91017645
19	2282777	1606298	−3.4816	2.6275	0.09868798	0.45508849	−4.1973630	−0.91017885
20	237967	−3932784	−0.74109	−1.2013	0.0986853501	0.4550898011	−4.197366237	−0.9101787855
21	6306379	9099768	−2.2087	1.7372	0.0986846123	0.4550898752	−4.197366968	−0.9101795943
22	8097245	−20058722	−1.0725	−1.6331	0.0986844093	0.4550898405	−4.197367716	−0.9101794241
23	8260177	42229518	−1.6674	1.0849	0.0986842476	0.4550899897	−4.197368040	−0.9101796673
24	70647235	−84750400	−1.6178	−1.9894	0.0986841815	0.4550898564	−4.197368083	−0.9101796616
25	−49306257	160983760	−1.4069	0.59098	0.0986841491	0.4550898673	−4.197368160	−0.9101797080
26	451798373	−284874718	−2.4037	−2.0703	0.0986841296	0.4550898600	−4.197368191	−0.9101797074
27	−527709591	453334314	−1.2884	0.19099	0.0986841228	0.4550898618	−4.197368211	−0.9101797164
28	2414035727	−589087984	−3.1839	−1.6188	0.0986841174	0.4550898607	−4.197368218	−0.9101797180
29	−3133922117	382030920	−1.2617	−0.14963	0.0986841158	0.4550898609	−4.197368223	−0.9101797197
30	11151899520	1113102974	−3.4637	−0.77741				

$\alpha_{3,n} = 4 + 3i$, $\alpha_{4,n} = -2i$, and the corresponding equation has the roots z_1, z_2, z_3, z_4. This occurs because of the presence of the double root $z_2 = 2$. For the same reason, all fifth order systems will be dependent.

A practical difficulty may arise with fixed decimal point machinery in that s_n may go off scale either on the right or the left. This will happen for sufficiently large n when $|z_1| \neq 1$. This difficulty may be overcome by a transformation $z' = z/r$, where $r = |\alpha_{M,n}|^{1/M}$, M being the order of a convergent process, n being "sufficiently" large. In our example we have $r = |\alpha_{2,n}|^{1/2}$. For n as small as three, $|s_n|$ would remain in a very satisfactory region.

AERODYNAMIC FORCES ON A SLOTTED FLAT PLATE*

By EDMUND PINNEY (*University of California, Berkeley*)

The derivation of the flat plate approximation to the aerodynamic forces exerted by an incompressible fluid on an airfoil is simple and is to be found in most textbooks. The present note treats the less simple case in which the airfoil contains slots oriented parallel to its length. This problem is of importance in the theory of suspension bridges where slotted roadbeds are used to cut down aerodynamically driven oscillations.

The spaces between the slots will be called "lanes". Let V be the velocity of flow, and α the angle of attack, assumed to be small. Let the plate occupy the interval $-b \leq x \leq b$, $y = 0$ of a rectangular coordinate system, and let $u(z)$, $v(z)$ be the velocity components of the fluid parallel to the x and y axes, respectively, at the point $z = x + iy$. Then, if powers of α higher than the first are neglected,

$$u(z) \to V, \qquad v(z) \to V\alpha \qquad \text{as } |z| \to \infty. \tag{1}$$

By two-dimensional potential theory, $v(z) + iu(z)$ is an analytic function of z away from the plate. Consider

$$v(z) + iu(z) = iV + V\alpha \prod_1^N{}_n (z - t_n b)^{1/2}(z - l_n b)^{-1/2} \tag{2}$$

where $x = t_n b$ and $x = l_n b$ are the trailing and leading edges, respectively, of the n-th lane, and N is the total number of lanes. The lanes are numbered from left to right so that

$$l_1 = -1, \qquad t_{n+1} > t_n, \qquad l_{n+1} > l_n, \qquad l_{n+1} > t_n, \qquad t_N = 1. \tag{3}$$

As $z \to \infty$, it is seen that (1) is satisfied.

In the n-th lane $l_n b < x < t_n b$, and by (2),

$$v(x \pm i0) + iu(x \pm i0) = iV \pm iV\alpha \prod_1^N{}_n |x - t_n b|^{1/2} |x - l_n b|^{-1/2}.$$

Therefore

$$u(x \pm i0) = V \pm V\alpha \prod_1^N{}_n |x - t_n b|^{1/2} |x - l_n b|^{-1/2}, \qquad v(x \pm i0) = 0. \tag{4}$$

*Received Sept. 22, 1947.

Therefore the velocity distribution of (2) satisfies the boundary conditions on the lanes. Also the velocity is finite when $z = t_n b$, so the Kutta condition is satisfied on the trailing edge of each lane.

By Bernoulli's theorem,

$$p + \frac{1}{2}\rho(u^2 + v^2) = \text{const.},$$

where p is the pressure and ρ is the density of the air. The quantities p and u are discontinuous on the lanes. Let p_+ , u_+ be their values on the upper side, and p_- , u_- their values on the lower side. Then

$$p_+ - p_- + \frac{1}{2}\rho(u_+^2 - u_-^2) = 0.$$

By (4), on the lanes,

$$p_+ - p_- + 2\rho V^2 \alpha \prod_1^N{}_n \mid x - t_n b \mid^{1/2} \mid x - l_n b \mid^{-1/2} = 0. \tag{5}$$

On the slots between the lanes, $p_+ - p_- = 0$, of course.

Let

$$\Phi(z) = \prod_1^N{}_n (z - t_n b)^{1/2}(z - l_n b)^{-1/2}/(z - b), \tag{6}$$

$$\begin{aligned} \phi(x) &= -\prod_1^N{}_n \mid x - t_n b \mid^{1/2} \mid x - l_n b \mid^{-1/2}/(b - x), \qquad \text{for} \quad x \text{ on a lane,} \\ &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad x \text{ on a slot.*} \end{aligned} \tag{7}$$

Then, anywhere on the roadbed,

$$p_+ - p_- = 2\rho V^2 \alpha (b - x)\phi(x). \tag{8}$$

We may now calculate the aerodynamic lift L and moment M from

$$L = \int_{-b}^{b} (p_+ - p_-)\, dx, \qquad M = \int_{-b}^{b} (p_+ - p_-)x\, dx.$$

It is possible to obtain these and higher aerodynamic moments, if desired, by evaluating

$$\int_{-b}^{b} [(p_+ - p_-)/(1 - rx/b)]\, dx$$

and expanding the result in powers of r.

If $0 < r < 1$, by (8),

$$\frac{b}{r}\int_{-b}^{b} \frac{p_+ - p_-}{x - b/r}\, dx = 2\pi\rho V^2 b\alpha\left\{\left(1 - \frac{1}{r}\right)\frac{b}{\pi r}\int_{-b}^{b} \phi(x)\frac{dx}{x - b/r} - \frac{1}{\pi r}\int_{-b}^{b} \phi(x)\, dx\right\}.$$

Now†

$$\frac{b}{\pi r}\int_{-b}^{b} \phi(x)\frac{dx}{x - b/r} = \frac{b}{r}\Phi\left(\frac{b}{r}\right)$$

*Pinney, E., Annals of Math. **47**, 221-232 (1946). See § 3.

†Pinney, E., loc. cit., formulas (1.1), (3.3).

Allowing r to approach zero, from (6), $(1/\pi)\int_{-b}^{b} \phi(x)\, dx = -1$, so

$$\int_{-b}^{b} (p_+ - p_-) \frac{dx}{1 - rx/b} = -2\pi\rho V^2 b\alpha \frac{1}{r}\left\{1 - (1 - r)\frac{b}{r}\Phi\left(\frac{b}{r}\right)\right\}.$$

Therefore, from (6),

$$\int_{-b}^{b} (p_+ - p_-) \frac{dx}{1 - rx/b} = -2\pi\rho V^2 b\alpha \frac{1}{r}\left\{1 - \prod_1^N{}_n \left(\frac{1 - rt_n}{1 - rl_n}\right)^{1/2}\right\}. \tag{9}$$

Now let

$$T_1 = \sum_1^N{}_n t_n, \qquad T_2 = \sum_1^N{}_n t_n^2, \qquad L_1 = \sum_1^N{}_n l_n, \qquad L_2 = \sum_1^N{}_n l_n^2. \tag{10}$$

Then

$$\prod_1^N{}_n (1 - rt_n)/(1 - rl_n) = 1 - r(T_1 - L_1) + \frac{1}{2}r^2[(T_1 - L_1)^2 - (T_2 - L_2)] + 0(r^3),$$

so

$$\prod_1^N{}_n (1 - rt_n)^{1/2}(1 - rl_n)^{-1/2} = 1 - \frac{1}{2}r(T_1 - L_1) + \frac{1}{8}r^2[(T_1 - L_1)^2 - 2(T_2 - L_2)] + 0(r^3).$$

This may be inserted into (9), expanding the right side in powers of r. Expanding the left side in powers of r and equating coefficients, we have

$$\begin{aligned} L &= -\pi\rho V^2 b(T_1 - L_1)\alpha, \\ M &= \frac{1}{4}\pi\rho V^2 b^2[(T_1 - L_1)^2 - 2(T_2 - L_2)]\alpha, \end{aligned} \tag{11}$$

for the aerodynamic lift and moment respectively.

Suppose the slots are symmetrically arranged about the centerline, and suppose ϵ is the fraction of the total roadbed width occupied by lanes. Then $T_2 = L_2$, $T_1 - L_1 = 2\epsilon$, so

$$L = -2\pi\rho V^2 b\epsilon\alpha, \qquad M = \pi\rho V^2 b^2 \epsilon^2 \alpha. \tag{12}$$

This proves the following theorem.

Theorem. A flat plate airfoil slotted symmetrically about its centerline has the same steady-state aerodynamic lift and moment as an unslotted flat plate of chord equal to the total chord of the slotted plate minus the total width of the slots.

NOTE ON THE THERMAL STRESSES IN A LONG CIRCULAR CYLINDER OF $m + 1$ CONCENTRIC MATERIALS*

By B. E. GATEWOOD (*AF Institute of Technology*)

It has been shown that the stresses and displacements in a long cylindrical body[1] can be expressed in terms of two functions U and V which satisfy the equations

$$\nabla^4 U = 0, \qquad \nabla^2 V = kT, \tag{1}$$

where T is the temperature change, and $k = Eq\alpha/(q - 1)$ with E Young's modulus, q the reciprocal of Poisson's ratio, and α the linear coefficient of thermal expansion. The stresses are as follows

$$\sigma_x = U_{yy} - V_{yy}, \qquad \sigma_y = U_{xx} - V_{xx}, \qquad \tau_{xy} = V_{xy} - U_{xy}$$
$$\tau_{zx} = 0, \qquad \tau_{yz} = 0, \qquad \sigma_z = Ee_z - kT + \nabla^2 U/q, \tag{2}$$

where the constant strain e_z is determined from the requirement that the total normal force on the cross-section of the cylinder be zero. The function U can be expressed[1] in terms of two analytic functions $\phi(z)$ and $H(z)$ as

$$U = z\,\bar{\phi}\,\overline{(z)} + \bar{z}\,\phi(z) + H(z) + \overline{H}\,\overline{(z)}, \tag{3}$$

and the function V may be taken as a particular integral of $\nabla^2 V = kT$ since the solution of $\nabla^2 V = 0$ can be included in $H(z)$.

The boundary conditions for the case of a circular cylinder composed of $m + 1$ different materials concentric to each other, each of which has different E, q, and α, can be obtained by taking the displacements and normal stresses continuous on the junction surfaces and the normal stress zero on the outside surface (let S_m be the inside material, S_{m-1} the adjoining material concentric to S_m, and S_0 the outside material; let C_0 of radius r_0 be the outside boundary and C_i of radius r_i the boundary between S_i and S_{i-1} with $j = 1, 2, \cdots, m$):

$$\phi_0(t) + t\,\bar{\phi}_0'\overline{(t)} + \bar{\psi}_0\overline{(t)} = [Y]_0, \text{ on } C_0, \tag{4}$$

$$(1 + M_i)\phi_i(t) + N_i\phi_{i-1}(t) + t\,\bar{\phi}_i'\overline{(t)} + \bar{\psi}_i\overline{(t)} = R_i t + [Y]_i, \text{ on } C_i \text{ with } j = 1, 2, \cdots, m, \tag{5}$$

$$M_i\phi_i(t) + (1 + N_i)\phi_{i-1}(t) + t\,\bar{\phi}_{i-1}'\overline{(t)} + \bar{\psi}_{i-1}\overline{(t)} = R_i t + [Y]_{i-1}, \text{ on } C_i, \tag{6}$$

where $z = t$ on C_i, $Y = V_x + iV_y$, $G = Eq/2(q + 1)$, $\psi(z) = H'(z)$, and

$$M_i = 4G_{i-1}(q_i - 1)/q_i(G_i - G_{i-1}), \qquad j = 1, 2, \cdots, m,$$
$$-N_i = 4G_i(q_{i-1} - 1)/q_{i-1}(G_i - G_{i-1}), \tag{7}$$
$$R_i = 2e_z G_i G_{i-1}(q_{i-1} - q_i)/q_i q_{i-1}(G_i - G_{i-1}).$$

*Received Sept. 22, 1947.

[1] B. E. Gatewood, *Thermal stresses in long cylindrical bodies*, Phil. Mag. (7) 32, 282-301 (1941).

Since the functions ϕ_i and ψ_i are analytic, they can be represented by a Laurent series in S_i $(j = 0, 1, \cdots, m-1)$ and a power series in S_m :

$$\phi_i(z) = \sum_{n=0}^{\infty} (a_{in} + ib_{in})z^n + \sum_{n=1}^{\infty} (c_{in} + id_{in})z^{-n},$$

$$\psi_i(z) = \sum_{n=0}^{\infty} (e_{in} + if_{in})z^n + \sum_{n=1}^{\infty} (g_{in} + ih_{in})z^{-n}, \tag{8}$$

with $c_{mn} = d_{mn} = g_{mn} = h_{mn} = 0$. Also, Y_i may be expressed in a series of powers of z such that on C_i $(j = 0, 1, \cdots, m)$

$$[Y]_i = \sum_{n=0}^{\infty} (A_{in} + iB_{in})t^n + \sum_{n=1}^{\infty} (C_{in} + iD_{in})t^{-n},$$

$$[Y]_{i-1} = \sum_{n=0}^{\infty} (E_{in} + iF_{in})t^n + \sum_{n=1}^{\infty} (G_{in} + iH_{in})t^{-n}. \tag{9}$$

If these expressions be substituted in the boundary conditions (4), (5), and (6) and the resulting expressions together with their conjugates be integrated over the respective circles by use of the Cauchy integral formula, then a system of equations for determining the desired analytic functions is obtained. In some cases the functions can be determined directly while in others they can be obtained by equating coefficients of like powers of z. For the general case the coefficients in Eq. (8) can be expressed in terms of the coefficients in Eq. (9) by means of recursion formulas. The coefficients $a_{i0} + ib_{i0}$, $e_{i0} + if_{i0}$, b_{i1}, and h_{i1} may be neglected since they contribute nothing to the stresses. a_{i1} and g_{i1} are determined by

$$\begin{aligned} &2a_{01} + g_{01}/r_0^2 = A_{01}, \qquad g_{m1} = 0, \\ &(2 + M_i)a_{i1} + N_i a_{i-1,1} + g_{i1}/r_i^2 = R_i + A_{i1}, \qquad j = 1, \cdots, m, \\ &M_i a_{i1} + (2 + N_i)a_{i-1,1} + g_{i-1,1}/r_i^2 = R_i + E_{i1}. \end{aligned} \tag{10}$$

Define $w_{in} = a_{in} + ib_{in}$, $x_{in} = c_{in} + id_{in}$, $y_{in} = e_{in} + if_{in}$, $z_{in} = g_{in} + ih_{in}$, $W_{in} = A_{in} + iB_{in}$, etc., whence w_{i2} and z_{i2} are given by

$$\begin{aligned} &w_{02} + \bar{z}_{02}/r_0^4 = W_{02}, \qquad z_{m2} = 0, \\ &(1 + M_i)w_{i2} + N_i w_{i-1,2} + \bar{z}_{i2}/r_i^4 = W_{i2}, \qquad j = 1, \cdots, m, \\ &M_i w_{i2} + (1 + N_i)w_{i-1,2} + \bar{z}_{i-1,2}/r_i^4 = Y_{i2}. \end{aligned} \tag{11}$$

The remaining coefficients w_{in}, $x_{i,n-2}$, $y_{i,n-2}$, and z_{in} $(n \geq 3)$ are given by $(p_i = r_{i+1}/r_i)$

$$\begin{aligned} &-M_{i+1}p_i^{2n}w_{i+1,n} + [1 + M_i - (1 + N_{i+1})p_i^{2n}]w_{in} + N_i w_{i-1,n} \\ &\quad - (n-2)r_i^{2-2n}\bar{x}_{i,n-2}(1 - p_i^2) = W_{in} - p_i^{2n}Y_{i+1,n}, \\ &-M_{i+1}\bar{x}_{i+1,n-2} + [(1 + M_i)p_i^{2n-4} - (1 + N_{i+1})]\bar{x}_{i,n-2} + N_i p_i^{2n-4}\bar{x}_{i-1,n-3} \\ &\quad + nr_{i+1}^{2n-2}(p_i^{-2} - 1)w_{in} = p_i^{2n-4}\bar{X}_{i,n-2} - \bar{Z}_{i+1,n-3}, \qquad j = 1, \cdots, m, \end{aligned}$$

$$x_{mn} = 0, \qquad z_{mn} = 0, \qquad \bar{z}_{0n} = r_0^{2n}[W_{0n} - w_{0n} + (n-2)r_0^{2-2n}\overline{x}_{0,n-2}],$$

$$\bar{z}_{0n} = r_1^{2n}[Y_{1n} - M_1 w_{1n} - (1+N_1)w_{0n} + (n-2)r_1^{2-2n}\overline{x}_{0,n-2}],$$

$$\bar{z}_{in} = r_i^{2n}[W_{in} - (1+M_i)w_{in} - N_i w_{i-1,n} + (n-2)r_i^{2-2n}\overline{x}_{i,n-2}],$$

$$y_{0,n-2} = r_0^{4-2n}[\overline{X}_{0,n-2} - nr_0^{2n-2}w_{0n} - \overline{x}_{0,n-2}],$$

$$y_{0,n-2} = r_1^{4-2n}[\overline{Z}_{1,n-2} - M_1\overline{x}_{1,n-2} - (1+N_1)\overline{x}_{0,n-2} - nr_1^{2n-2}w_{0n}],$$

$$y_{i,n-2} = r_i^{4-2n}[\overline{X}_{i,n-2} - (1+M_i)\overline{x}_{i,n-2} - N_i\overline{x}_{i-1,n-2} - nr_i^{2n-2}w_{in}].$$

For the case of the temperature a function of the radius the functions become very simple with the series in Eqs. (8) and (9) reducing to one term. With the temperature as $T_i(r)$ in S_i, the particular integral of $\nabla^2 V = kT$ yields

$$z\,\overline{Y}_m = k_m \int_0^r rT_m(r)\,dr,$$

$$z\,\overline{Y}_i = k_m \int_0^{r_m} rT_m(r)\,dr + k_{m-1}\int_{r_m}^{r_{m-1}} rT_{m-1}(r)\,dr + \cdots + k_i \int_{r_{i+1}}^{r} rT_i(r)\,dr,$$

whence, using Eq. (9), $[Y]_i = L_i t = A_{i1}t = [Y]_{i-1} = E_{i1}t$ with $r_i^2 L_i = [z\overline{Y}_i]_{r=r_i}$. Further, $\phi_i(z) = a_{i1}z$ and $\psi_i(z) = g_{i1}/z$, where a_{i1} and g_{i1} are given by Eq. (10). The function U is $U_i = a_{i1}r^2 + g_{i1}\log r$ and the radial and tangential stresses in polar coordinates are

$$\sigma_{ri} = 2a_{i1} + (g_{i1} - z\overline{Y}_i)/r^2, \qquad \tau_{r\theta i} = 0,$$

$$\sigma_{\theta i} = 2a_{i1} - (g_{i1} - z\overline{Y}_i)/r^2 - k_i T_i\,,$$

$$\sigma_{zi} = E_i e_z - k_i T_i + 4a_{i1}/q_i\,.$$

ON THIRD-ORDER CORRELATION AND VORTICITY IN ISOTROPIC TURBULENCE*

By F. N. FRENKIEL (*Cornell University*)

The fundamental equation of the propagation of the correlation function obtained by Th. von Kármán and L. Howarth[1] for isotropic turbulence in an incompressible viscous fluid, is written

$$\frac{\partial(\overline{u'^2}f)}{\partial t} + 2(\overline{u'^2})^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4}{r}h\right) = 2\nu\overline{u'^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right). \tag{1}$$

*Received Nov. 19, 1947.

[1]Th. von Kármán and L. Howarth. *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. London (A) **164**, 192 (1938).

For small distances r, the longitudinal correlation function f may be represented by the relation†

$$f = 1 + \frac{1}{2} r^2 f''(0) + \frac{1}{24} r^4 f^{IV}(0)$$

and using the two relations

$$\lambda^{-2} = -f''(0) \qquad \text{and} \qquad \lambda_\omega^{-2} = \frac{7}{15} \lambda^2 f^{IV}(0)$$

where λ is Taylor's dimension of the smallest eddies, there will be obtained for small r the following expression:

$$f = 1 - \frac{r^2}{2\lambda^2} + \frac{5}{56} \frac{r^4}{\lambda^2 \lambda_\omega^2} \tag{2}$$

Applying now the well-known relation[1]

$$\frac{1}{\overline{u'^2}} \frac{d\overline{u'^2}}{dt} = -10 \frac{\nu}{\lambda^2} \tag{3}$$

and multiplying by λ^3/r^2, one has, for small r,

$$\frac{d\lambda}{dt} + 5 \frac{\nu}{\lambda} \left(1 - \frac{\lambda^2}{\lambda_\omega^2}\right) + 2 \frac{\lambda^3 (\overline{u'^2})^{1/2}}{r^2} \left(\frac{\partial h}{\partial r} + \frac{4}{r} h\right) = 0 \tag{4}$$

This equation gives the law of the propagation of the dimension λ. It may be also written in the following form

$$\frac{\partial h}{\partial r} + \frac{4}{r} h = \left[\frac{5}{2}\left(\frac{\lambda^2}{\lambda_\omega^2} - 1\right) - \frac{1}{4\nu} \frac{d\lambda^2}{dt}\right] \frac{\nu r^2}{\lambda^4 (u'^2)^{1/2}} \tag{4a}$$

and after integrating this differential equation we find

$$h = \left[\frac{5}{14}\left(\frac{\lambda^2}{\lambda_\omega^2} - 1\right) - \frac{1}{28} \frac{1}{\nu} \frac{d\lambda^2}{dt}\right] \frac{1}{N_\lambda} \left(\frac{r}{\lambda}\right)^3 \tag{5}$$

where

$$N_\lambda = \frac{\lambda (\overline{u'^2})^{1/2}}{\nu}$$

is the number of Reynolds of microturbulence.

Let us consider the decay of isotropic turbulence and assume with von Kármán the case of "large" Reynolds number of turbulence, i.e. the case when the correlation functions $f(r, t)$ and $h(r, t)$ are independent of the viscosity ν, except for small r. In addition, von Kármán assumes that the correlation curves preserve their shape and that only a scale L changes. As a result the relation[1]

$$\frac{(\overline{u'^2})^{1/2} \lambda^2}{L\nu} = A \tag{6}$$

is found, where A is a numerical constant and L a dimension representing the scale of turbulence.

†To avoid confusion between values at an initial moment t_0 which appears later and values at the origin, we shall use in this paper $f''(0)$ for f'' at $r = 0$ instead of f_0'', $h'''(0)$ instead of h_0''', and so on.

L. G. Loitsianskii[2] shows, that when the rate of decrease of f, h and $\partial f/\partial r$ exceeds the rate of increase of r^4, there exists the following theorem of the conservation of the disturbance moment:

$$\overline{u'^2} \int_0^\infty s^4 f(s)\, ds = \overline{u'^2}\, L^{*5} = \text{const.} \tag{7}$$

As the disturbance length L^* is given as a function of the fourth moment of f and as the shape of this curve is assumed to be preserved for large r, we may use L^* in place of L, assuming that L^*/L is constant. Thus the equation (6) becomes

$$\frac{(\overline{u'^2})^{1/2}\lambda^2}{L^*\nu} = A_1 \tag{6a}$$

where A_1 is a new numerical constant.

After eliminating L^* between (6a) and (7) it is found that

$$\frac{(\overline{u'^2})^{7/10}\lambda^2}{\nu} = \frac{1}{C} \tag{8}$$

with C a numerical constant. Eliminating now λ^2/ν between the last relation and the equation (3) and integrating, we find

$$(\overline{u'^2})^{-7/10} - (\overline{u_0'^2})^{-7/10} = 7C(t - t_0) \tag{9}$$

where $\overline{u'^2} = \overline{u_0'^2}$ at a chosen initial moment $t = t_0$. C is found as a function of the initial values $\overline{u_0'^2}$ and λ_0 using equation (8), and as a result the following equation of decay is obtained:

$$\overline{u'^2} = \overline{u_0'^2}\left[7\,\frac{\nu}{\lambda_0^2}\,(t - t_0) + 1\right]^{-10/7} \tag{10}$$

It will also be found that

$$L^* = L_0^*\left[7\,\frac{\nu}{\lambda_0^2}\,(t - t_0) + 1\right]^{2/7} \tag{11}$$

and

$$\lambda^2 = \lambda_0^2\left[7\,\frac{\nu}{\lambda_0^2}\,(t - t_0) + 1\right] \tag{12}$$

or

$$\lambda^2 = \lambda_0^2 + 7\nu(t - t_0) \tag{12a}$$

In another paper[3] these results are compared with experimental measurements of the decay of turbulence, and the agreement with equations (10) and (12) seems quite satisfactory.

The variation of the Reynolds Number of microturbulence is easily found using equations (10) and (12) and is written

$$N_\lambda = N_{\lambda,0}\left[7\,\frac{\nu}{\lambda_0^2}\,(t - t_0) + 1\right]^{-3/14} \tag{13}$$

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[2]L. G. Loitsianskii. *Some basic laws of isotropic turbulent flow*, N. A. C. A., T. M. 1079, 1945 (Translated from Rep. No. 440 of Central Aero-Hydrodynamical Institute, Moscow, 1939).

[3]F. N. Frenkiel. *The decay of isotropic turbulence*, (to be published).

For the decay of vorticity it will be found

$$\overline{\omega^2} = \overline{\omega_0^2}\left[7\,\frac{\nu}{\lambda_0^2}\,(t - t_0) + 1\right]^{-17/7} \tag{14}$$

with $\overline{\omega^2} = 15\overline{u'^2}/\lambda^2$, the value of the mean square of the vorticity, $\overline{\omega_0^2}$ being its value at the initial moment t_0 .

Differentiating equation (12a) it will be found

$$\frac{1}{\nu}\frac{d\lambda^2}{dt} = 7 \tag{15}$$

which combined with equation (5) gives for the third order correlation coefficient the expression

$$h = \left[\frac{5}{14}\frac{\lambda^2}{\lambda_\omega^2} - \frac{17}{28}\right]\frac{1}{N_\lambda}\left(\frac{r}{\lambda}\right)^3 \tag{16}$$

In this relation the third order correlation coefficient h at the moment t is given as a function of λ, λ_ω and N_λ at the same moment t.

Equation (16) represents a parabola of the third order to which the triple correlation curve $h(r)$ is tangent at its origin.

The last equation gives

$$h'''(0) = \left[\frac{5}{14}\frac{\lambda^2}{\lambda_\omega^2} - \frac{17}{28}\right]\frac{6}{\lambda^3 N_\lambda} \tag{17}$$

It should be recalled that

$$\frac{\lambda^2}{\lambda_\omega^2} = \frac{7}{15}\frac{f^{IV}(0)}{[f''(0)]^2}$$

depends on the shape of the correlation curve. Using the notations of Batchelor and Townsend[4] $S = -\lambda^3 k'''(0)$; $G = \lambda^4 f^{IV}(0) = f^{IV}(0)/[f''(0)]^2$ and remarking that for isotropic turbulence[1] $k = -2h$ and $k'''(0) = -2h'''(0)$ it will be found from (17) that

$$G = \frac{51}{14} + \frac{1}{2}N_\lambda S \tag{18}$$

giving a relation between G, S and N_λ at the same moment t.

Consider the case of decay of isotropic turbulence and assume that $G = G_0$ is constant during the decay. This assumption agrees with the results of the experiments of Batchelor and Townsend.[4] The constant value of G_0 depends upon the experimental conditions and may be different in each experiment, depending on the shape of the correlation curve. Thus during the decay

$$S = \frac{2}{N_\lambda}\left(G_0 - \frac{51}{14}\right) \tag{19}$$

and at the initial moment t_0

$$S = S_0\left[7\,\frac{\nu}{\lambda_0^2}\,(t - t_0) + 1\right]^{3/14} \tag{20}$$

[4]G. K. Batchelor and A. A. Townsend. *Decay of vorticity in isotropic turbulence*, Proc. Roy. Soc. London (A), 190, 534 (1947).

with
$$S_0 = \frac{2}{N_{\lambda,0}}\left[G_0 - \frac{51}{14}\right] \tag{21}$$

The decay of the vorticity is given by the relation[1]

$$\frac{d\overline{\omega^2}}{dt} = 70h'''(0)\cdot(\overline{u'^2})^{3/2} - 10\nu\frac{\overline{\omega^2}}{\overline{\lambda_\omega^2}} \tag{22}$$

and it will easily be found, using the relations given before, that

$$\frac{d\overline{\omega^2}}{dt} = \frac{7}{3(15)^{1/2}}(\overline{\omega_0^2})^{3/34}\left[S_0 - 2\frac{G_0}{N_{\lambda,0}}\right](\overline{\omega^2})^{24/17} \tag{23}$$

In the decay of vorticity, the rate of production of vorticity by diffusive stretching of the vortex tubes and the rate of dissipation of vorticity by viscosity are each proportional to $(\overline{\omega^2})^{24/17}$. The ratio of the production of vorticity to its dissipation is constant and equal to

$$\frac{SN_\lambda}{2G} = 1 - \frac{51}{14}\frac{1}{G_0} \tag{24}$$

The equations (23) and (21) give

$$\frac{d\overline{\omega^2}}{dt} = -\frac{17}{(15)^{1/2}}\frac{(\overline{\omega_0^2})^{3/34}}{N_{\lambda,0}}(\overline{\omega^2})^{24/17} \tag{25}$$

which could be found directly by differentiating equation (14).

There exist some differences between the conclusions drawn here for isotropic turbulence at "large" Reynolds number of turbulence and the conclusions of Batchelor and Townsend. In particular, in the present paper it is found that the decay of vorticity is proportional to $(\overline{\omega^2})^{24/17}$ and not to $(\overline{\omega^2})^{3/2}$. The factor G is considered to be constant as in the Batchelor-Townsend paper but N_λ and S are variable. For the case studied here, G is given as function of S by the equation (18), which is of the same form as that found by Batchelor and Townsend but instead of 30/7 the constant is equal to 51/14.

The experimental results of Batchelor and Townsend[4] seem to agree only very roughly with these relations. However, if account is taken of probable inaccuracies in the experimental data, such as the imperfect isotropy behind grids and errors due to finite lengths of hot-wires (especially as this concerns measurements of λ), it appears that the agreement may be satisfactory after all.

BOOK REVIEWS

Relaxation methods in theoretical physics. By R. V. Southwell. Oxford, at the Clarendon Press, 1946. vii + 248 pp. $7.00.

The relaxation method of approximate numerical solution of systems of equations was first introduced in 1936 by R. V. Southwell in connection with the solution of engineering problems arising in the field of structural design. In questions of the loading and deflection of complex structural frames, one is struck by the inherent precision of the conventional analytical methods of attack and the inherent lack of precision in the given data defining a problem. A number of people devised numerical schemes of calculation by which the precision of the answer could be made as good as desired and thus made comparable with the given data. Several of these methods involved successive relaxation of constraints. To Southwell

however, belongs the credit of realizing the great generality of this method of approach. In steps, he and his team of computers have solved more and more complex problems.

In the book "Relaxation Methods in Engineering Science," Southwell has presented in some detail, problems involving systems of algebraic equations and ordinary differential equations, including eigenvalue problems, which arise in engineering structures and vibrations problems.

In the present volume, the treatment of problems by the relaxation method is extended to include partial differential equations from various branches of classical physics. All of the problems are included in the general form

$$\frac{\partial}{\partial x}\left(\psi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\psi\frac{\partial\phi}{\partial y}\right)=Z, \tag{1}$$

where

$$\psi=\psi(x, y, \phi, \phi_x, \phi_y),\ Z=Z(x, y, \phi, \phi_x, \phi_y). \tag{2}$$

The first chapter is devoted to a discussion of general ideas and formulas. The relaxation method is explained and the finite difference expressions for differentials of various orders are derived. Partial differentials are then treated, and in particular the finite difference approximation to the Laplace operator is derived for triangular, square and hexagonal nets. Numerical integration and interpolation are also briefly treated in this chapter.

The remainder of the book, chapters II through VI are devoted to successively more difficult cases of (1) and (2). The added difficulties are of two classes: (a) more difficult boundary conditions, and (b) more complex forms of (1) and (2).

In chapter II, Equation (1) with $\psi = 1$, $Z = Z(x, y)$ is discussed in detail. (No significant numerical simplification results if $Z = 0$.) The relaxation pattern, the basic idea of relaxation, block relaxation for speedier approach to the solution, advance to a finer net, and the physical significance of these techniques in terms of the "tensioned net" are all carefully presented. The chapter concludes with several torsion problems, some of which are left for the reader to try.

In chapter II, the boundary conditions are specified values of the wanted function on specified rectangular boundaries. In chapter III, the treatment is broadened to include specified normal gradients and irregular boundaries. Several problems with known analytical solutions are solved to illustrate the accuracy attained with relatively course nets and then several much more difficult problems involving the torsion of irregular shafts and magnetic fields in electric machines are solved to illustrate the power of the method.

Chapter IV deals with general questions of conformal transformation. The numerical determination of the real and imaginary parts of the complex function required to transform any given domain into a circle or rectangle is considered. A treatment for suitable removal of the logarithmic singularity at the center of the circle is given as is the determination of the complex conjugate of a function. It is shown that the most accurate determination of a complex conjugate generally involves a separate relaxation for each.

Cases of (1) with $\psi = \psi(r, y)$, of which the Laplace operator in polar coordinates (r, z) is a typical example, are treated by the use of relaxation patterns which vary from point to point. Chapter V presents the necessary theory and gives a considerable number of illustrative problems from the fields of elasticity, heat conduction, and lubrication. This chapter concludes with two problems in which ψ is a function of ϕ and its derivatives as well as x, y. A lubrication problem with variable viscosity and the flow of a gas through a nozzle are the illustrative problems treated in detail.

In the final chapter (VI) "Problems Involving Boundaries or Interface Not Initially Known" are treated. Usually boundary conditions consist of two parts, a given boundary and a given relation between the function and its normal derivative along this boundary. Instead, one may choose two other suitable conditions say the value of the function and the value of its normal derivative but omitting the shape of the boundary itself. Such problems arise in a natural way in many physical problems. The problems discussed involving plastic torsion, percolation of water through soils, the waterfall, and the wake behind a cylinder are all cases in which analytical methods have given few or no solutions.

No exaggerated claims are made for the ease by which the problems can be solved by the relaxation method. It is in fact pointed out that problems like those of chapters V and VI especially involve considerable labor of a type which does not appeal to many persons. It is correctly noted however that problems

of these and even more difficult types (not yet released from wartime publication restrictions) can be solved by the relaxation method, whereas they are solvable in no other way at present.

Throughout the book, it is emphasized that the only way to appreciate the significance of the relaxation method of numerical solution of partial differential equations of mathematical physics is to do problems. Thus each chapter presents the physical theory and finally accepted answer to a number of problems, leaving the numerical work as exercise for the reader.

Numerous tables of finite difference formulas for differentiation and integration are given in 10 pages of appendix.

HOWARD W. EMMONS

Conduction of heat in solids. By H. S. Carslaw and J. C. Jaeger. Oxford at the Clarendon Press, 1947. viii + 386 pp. $8.00.

This book is a very extensively revised and enlarged version of the senior author's "Introduction to the Mathematical Theory of the Conduction of Heat in Solids" which appeared in 1921. It treats very thoroughly the best available mathematical techniques for the analysis of heat conduction and includes a large number of solutions to specific problems.

The first chapter presents a discussion of the basic partial differential equation, and the mathematical formulation of initial and boundary conditions for typical systems. In the following eight chapters the Fourier methods of analysis are applied to determine transient temperature distributions under various conditions in infinite and semi-infinite solids, slabs, rods, rectangles, rectangular parallelepipeds, circular cylinders, wedges, spheres and cones. In Chapter X, the method of sources and sinks and the method of images are developed and applied to problems not easily solvable by the methods of the preceding chapters. Much, though by no means all of the material in these ten chapters was contained in the earlier book; the remainder of the book is largely new.

The next four chapters present a clearly written discussion of the Laplace transform and a thorough treatment of its application to the solution of heat conduction problems. This very powerful method, which was developed after the appearance of the earlier book, replaces the "method of contour integrals" discussed therein, and furnishes solutions to many complex problems which do not yield readily to the more elementary techniques. Many of the original results obtained by the authors are reproduced or presented for the first time in these chapters. The Laplace transform is also used to obtain the Green's functions for heat conduction in solids of various typical geometries. A final chapter includes for completeness a very brief treatment of steady state heat conduction.

Many graphs and tables not contained in the earlier book and useful in making calculations for engineering applications are included for ready reference. A very brief discussion of the available numerical and graphical methods is also given. Many references to the literature are made, but there has been no attempt to compile an exhaustive bibliography. The theory developed is a linear one, i.e. no attempt has been made to treat the problem of variable thermal properties. A minor defect is that the table of contents and the index are possibly over-concise, so that a page-by-page search is sometimes necessary to locate a desired result. The book as a whole is a clearly written, complete and excellent treatise on the subject.

S. A. SCHAAF

Tables of supersonic flow around yawing cones. By the staff of the Computing Section, Center of Analysis, Massachusetts Institute of Technology. Under the direction of Zdenek Kopal. Cambridge, Massachusetts, 1947. xviii + 321 pp. $5.00.

In the introduction to this volume, the theory governing the supersonic flow of a gas past a slightly yawed cone is developed. In this development, the Taylor-Maccoll solution for a symmetrically oriented cone forms the zero order terms of a perturbation expansion with regard to the yaw angle ϵ. The equations relating to the terms of order ϵ are derived. The original detailed theory which has not been previously published is said to be forthcoming in a periodical.

The problem reduces to the determination of the coefficient of ϵ in the perturbation development and the body of the book consists of a compilation of the numerically obtained results. The range of cone angles is $5° \leq \vartheta_s \leq 50°$. As in volume I (the corresponding tables for the unyawed cone) results are obtained for a large number of wave angles corresponding to each cone angle.

G. F. CARRIER

Problème général de la stabilité du mouvement. By M. A. Liapounoff. Princeton University Press, Princeton, 1947. 273 pp. $3.50.

Number seventeen in Annals of Mathematical Studies, this book is a translation from the Russian. Published in Russian in 1892 by the Mathematical Society of Kharkow, the French translation appeared in Annales de la Faculté des Sciences de Toulouse, 2nd ser., vol. 9 (1907).

The book is divided into three chapters, the first of which is labeled Preliminary Analysis; it includes definition and discussion of the general problem of stability. Chapter I also contains a discussion of systems of linear differential equations and differential equations of the disturbing motion. Chapter II is entitled Study of Steady Motion and it includes more specific treatment of the differential equations of the disturbing motion for various cases of steady motion. Chapter III is concerned with the Study of Periodic Motion and includes analysis of the characteristic equation as well as the differential equations of the disturbing motion.

This work encompasses the classical treatment of stability for continuous motion of disturbance. There is not any discussion of dynamical processes subject to discontinuous impulses.

R. Truell

Theory and application of Mathieu functions. By N. W. McLachlan. London: At the Clarendon Press, 1947. xii + 401 pp. $12.50.

In the first two-thirds of this treatise the developments of the solutions of the Mathieu equation are carried out and various techniques for the evaluation of the necessary parameters are given in some detail. These techniques, both numerical and analytic, are amply illustrated by appropriate numerical examples, and their ranges of validity are indicated. The solutions are developed in the form $e^{\mu z}\varphi(z)$ where φ is found as a trigonometric series, in the form of Maclaurin series, and in series of Bessel functions. The eigenvalue problem associated with the Mathieu equation is treated. Orthogonality conditions and several integral identities are established. The asymptotic representations and zeros of Mathieu functions are discussed. The Hill equation is briefly treated.

In the second part, various physical problems are solved in terms of Mathieu functions and the results are interpreted. Loud speaker distortion, the oscillations of an elastic column under periodic end thrust, the vibration of a membrane with an elliptical boundary, the viscous flow past an elliptic cylinder, and other problems, are used to illustrate the large class of problems in which Mathieu functions arise.

G. F. Carrier

Tables of spherical Bessel functions. Prepared by the Mathematical Tables Project, National Bureau of Standards. Vol. II. Columbia University Press, New York, 1947. xx + 328 pp. $7.50.

In this continuation of Vol. I of the same work, the function $(\pi/2x)^{1/2}J(x)$ is tabulated for the half-odd-integer values $29/2 \leq |\nu| \leq 61/2$. In the range $29/2 \leq |\nu| \leq 43/2$, the argument x ranges from 0 to 10 in steps of .01 and from 10 to 25 in steps of .10. For $45/2 \leq |\nu| \leq 61/2$ the range of x is 10 to 25 in steps of .10. For $\frac{1}{2} \leq \nu \leq 61/2$ and for $-29/2 \geq \nu \geq -61/2$, the function $2^{\nu}\Gamma(\nu - 1)J(x)/x^{\nu}$ is tabulated for $0 \leq x \leq 25$ at increments of .1. The number of significant figures is at least seven. Finally, some of the zeros, maxima, and minima, of the half order Bessel functions are tabulated for $-29/2 \leq \nu \leq 45/2$.

G. F. Carrier

Vector and tensor analysis. By Louis Brand. John Wiley & Sons, Inc., New York, 1947. xvi + 439 pp. $5.50.

This text contains a very good introduction to the classical theory of vectors, motors, dyads, tensors, and quaternions and some of their applications to geometry and physics. Throughout the text, the main emphasis is on developing the student's manipulative ability for dealing with these powerful tools. This is accomplished by offering very complete explanations of techniques and using numerous illustrative problems throughout the text.

The first section deals with Gibbs' Vector Analysis (Chapters I, III, V, VI). The dot, cross, triple scaler, etc. products and reciprocal basis are introduced in Chapter I. As applications, the author offers proofs of various theorems in classical plane projective geometry and vector interpretations of homogeneous, line, and plane coordinates. Following the introduction of vector differentiation and integration (Chapter III), the author treats the theory of curves (the Frenet formulas and applications). As a further illustration of vectorial methods over one dimensional fields, the kinematics of a particle is discussed. The portion of the text which treats vector fields of two and three dimensions (Chapters V, VI) discusses such topics as the gradient, the curl, the divergence, vector derivative identities, curvilinear coordinates, surface vectors, the Green and Stokes theorems and some elementary potential theory. It is the reviewer's opinion that the author uses the cross product in several instances where use of the dot product would be more advantageous. Thus, in deriving the Frenet formulas, the author works with the cross product. Here, the dot product furnishes an easier derivation and one that can be generalized to n-dimensional manifolds.

The algebra of motors is studied in Chapter II. This work is used in only one application (Chapter III, pp. 117). Thus, this topic may be included or excluded in any course based on this text without affecting the continuity of the remaining material.

Tensors and their applications form the basis of Chapters IV, VII, VIII, IX. The classical Gibbs' theory of dyadic algebra is discussed in Chapter IV. Topics such as conjugate dyadics, decomposition of dyadics, symmetry, invariants of dyadics, relation between dyadics and matrices are treated. Some use of these ideas is made in Chapters V and VI. In the main, the use of dyadics in these last mentioned chapters serves to generalize the Green and Stokes vector theorems. However, fairly extensive use of dyadics is made in the treatment of Hydrodynamics (Chapter VII) and Surface Geometry (Chapter VIII). The section on Hydrodynamics discusses the Euler and Lagrange equations, the Helmholtz equations, and the Kutta-Joukowsky formula. In the Chapter on Surface Geometry, the author treats curves on a surface by means of the Darboux moving trihedral. Geodesic curvature, geodesic torsion, normal curvature, lines of curvature, asymptotic lines are among the many topics introduced. Further, the first and second fundamental forms, geodesics, the mean and total curvature, the Gauss-Bonnet formula, and the Gauss-Codazzi relations are all discussed in some detail. Chapter IX is concerned with Tensor Analysis. Transformation laws of vectors and tensors, multiplication and contraction of tensors, tensor densities, the permutation symbol, and the generalized Kronecker tensors are examined. By expanding geometric objects (dyads, triads, etc.) in terms of their tensor components and a system of base vectors and by determining the differentiation law of the base vectors, the author introduces covariant differentiation. The Riemann-Christoffel and Ricci tensors are discussed. The reviewer would like to make two criticisms of the work in this section. First, the use of dyads in the section on Surface Geometry (Chapter VIII) needlessly increases the amount of technique which students must develop and, further, detracts from the unity of the text. The same results can be obtained by the proper use of tensors (components of the dyads). Secondly, in deriving the Kutta-Joukowsky formula, the author passes from vectors to complex numbers by a vague equivalence scheme. It is known (see A. Wundheiler: *Are Complex Numbers Vectors?*: Bull. of Am. Math. Soc., Vol. 46, No. 1, Jan. 1940, pp. 57) that "complex numbers are not isomorphic with vectors but are vectors in a metric plane with a distinguished direction." Thus, it appears desirable to show clearly the relation between complex numbers and vectors or else to treat complex numbers independently of vectors.

In the final Chapter X of the text, Quaternion Algebra is developed. Some applications are given to rigid body motions.

The author has presented in an interesting and readable manner several methods of treating geometric quantities. As a result, this book should form a valuable addition to the existing texts on Vector and Tensor methods.

N. Coburn

Sequential analysis. By Abraham Wald. John Wiley & Sons, Inc., New York, 1947. xii + 212 pp. $4.00.

One of the most notable recent developments in statistical theory is sequential analysis, a sampling procedure in which the size of the sample is not fixed in advance, but depends on the results of the successive observations. The basic results are due to Professor Wald, and have been published in two papers in the Annals of Mathematical Statistics; this book is essentially an expanded and somewhat reorganized version of Professor Wald's papers, together with some additional material.

The book consists of three parts and an appendix. Part I contains background material and most of the theory, including a discussion of the sequential probability ratio test, the most important sequential procedure. Part II is a detailed description of sequential procedures for testing certain hypotheses of practical importance; the hypotheses treated concern the parameters of binomial and normal populations. Part III is a brief discussion of some possible sequential approaches to the problem of multi-valued decisions and estimation. The appendix contains those parts of the theory which require some advanced mathematics; the rest of the book is not beyond the understanding of one acquainted with elementary calculus.

Two features of the sequential method deserve especial mention. 1) The sequential procedure requires on the average fewer observations than comparable fixed-sample-size procedures, the saving in some cases of practical importance being as much as 50%. 2) Sequential theory is much simpler than that for samples of fixed size. The book is a clear presentation of an important topic, and should be invaluable to practical statisticians.

DAVID BLACKWELL

Guide to the literature of mathematics and physics including related work on engineering science. By Nathan Grier Parke III. McGraw-Hill Book Co., Inc., New York and London, 1947. xv + 205 pp. $5.00.

This "Guide" is planned for the "large group of people who cannot maintain an independent awareness of the mathematical and physical reference literature but who can make effective use of a classified guide. This group includes scientists, engineers, librarians, and students . . . Often the techniques that overcome obstacles in one field can be effectively used in another. For example, linear electrical networks are the most intensively studied vibration systems. By means of electromechanical analogues, all these results from the electrical field are applicable to the mechanical field." The bibliography of approximately 1800 entries, classified under some 150 headings, is preceded by over 70 pages of detailed discussion of the principles of reading and study, self-directed education, literature search, and too little on the use of periodicals. Suggestions are offered on problems of library technique and use.

As the intended audience is so inhomogeneous, both in interests and in level of accomplishment, the author has necessarily assumed almost complete lack of knowledge on the part of the reader. An experienced worker, however, would do well to read all of the, what might appear to him as a somewhat pedantic and overdetailed, discussion for the valuable hints it contains. Unfortunately a difficulty arises in the use of the bibliography whose primary functions should be to tell the reader about the essential treatises in each field, books which bring the information up to date, and also periodicals in which reviews and research reports are published. In some cases, the author has done this carefully, but in many classifications, the research worker or engineer who is branching out, possibly temporarily, into an unfamiliar field is confronted with too many books about which too little is said for him to make an intelligent selection.

In its present form, the "Guide" is most useful to those who have an elementary understanding of the topics considered and who require detailed or advanced knowledge. It is to be hoped that frequent revisions or supplements will be issued to keep the "Guide" valuable. As an example, illustrating this necessity, *Applied Mechanics Reviews* will soon be published as a periodical covering the current world literature in many of the classifications. Obviously, no mention of this valuable addition to the review journals can appear in the present "Guide."

D. C. DRUCKER

Table of the Bessel functions $J_0(z)$ and $J_1(z)$ for complex arguments. Prepared by the Mathematical Tables Project, National Bureau of Standards. Second edition. Columbia University Press, New York, 1947. xliv + 403 pp. $7.50.

The preface states that since the publication of the first edition of these tables [reviewed in this Quarterly, *2*, 276 (1944)] no errors have been reported in the tabular material. Accordingly, this material could be reproduced from the negatives which were used for the first edition. Some corrections and revisions were made in the Introduction, however, which now contains the relations between the tabulated functions on the 45° ray and the *ber* and *bei* functions.

W. PRAGER

Nomograms of complex hyperbolic functions. By Jørgen Rybner. Jul. Gjellerups Forlag, Copenhagen, 1947. xxxvi + 54 pp. Price Danish Kroner 24.00.

This table of nomograms includes an introduction containing a note on the history of real and complex hyperbolic functions and notation, earlier tables and graphical representations of complex hyperbolic functions, the use of nomograms for: $\cosh (b + ja) = p + jq$, $\sinh (b + ja) = p + jq$, $\tanh (b + ja) = r \angle \theta = re^{i\theta}$, $x + jy = r \angle \theta = re^{i\theta}$ and $R \angle \alpha = 1 + r \angle \phi$ as well as the use of nomograms for reflection losses and reflection phase shift and for resonance circuits and filters. Approximately ten pages of formulas for circular and hyperbolic functions include real functions, gudermannian angle, integrals, differentials, series developments, functions of imaginary argument, functions of complex argument and inverse complex hyperbolic functions.

A table of multiples of $\pi/2$ from 1 to 100 is present as well as some formulas for four terminal networks and transmission lines.

All of the material mentioned above is contained in thirty five pages (8½ x 12). In addition there are fifty pages of nomograms as follows:

$\cosh (b + ja) = p + jq$ 13 nomograms covering the range $b = 0$ to $b = 4.0$ nepers
$\sinh (b + ja) = p + jq$ 13 nomograms covering the range $b = 0$ to $b = 4.0$ nepers
$\tanh (b + ja) = r \angle \theta$ 16 nomograms covering the range $b = 0$ to 2.0 nepers

Eight additional nomograms of elementary functions related chiefly with transmission line and network calculations.

This book of nomograms has been prepared with unusual care and accuracy, and its contents should be of considerable value to electrical and acoustical engineers and possibly to physicists. Any extensive numerical calculations involving impedances or reflection and transmission coefficients can be facilitated by means of these nomograms.

ROHN TRUELL

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